Data-flow Parallel Implementation with OmpSs of ILUPACK

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Motivation and Introduction

- We consider the **efficient** solution of linear systems

$$Ax = b$$

  where
  - $A \in \mathbb{R}^{n,n}$ is a **large** and **sparse** SPD coefficient matrix
  - $x \in \mathbb{R}^n$ is the sought-after solution
  - $b \in \mathbb{R}^n$ is a given r.h.s. vector
- The solution of this problem
  - arises in very many large-scale application problems
  - is frequently the most time-consuming part of the computation

Motivation and Introduction

Two main approaches to “attack” the problem:

- Direct methods (sparse Cholesky, sparse LU, etc.) may fail due to excessive fill-in of the factors
- Iterative methods (CG, GMRES, etc.) may present slow convergence or even divergence

Convergence of iterative methods can be accelerated via preconditioning:

- Incomplete LU factorization
- Sparse approximate inverse
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We propose to solve $Ax = b$ **iteratively** using

- Preconditioned Conjugate Gradient (PCG)
  - Among the best iterative approaches to solve SPD linear systems
- ILUPACK preconditioning techniques
  - Incomplete Cholesky (IC) factorization-based preconditioners
    $\rightarrow$ Multilevel IC (MIC) preconditioners: **Inverse-based** approach
  - Successful for a wide range of large-scale application problems of scientific interest with up to million of equations

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- The large scale of the problems faced by ILUPACK makes it evident the need to reduce the time-to-solution via parallel computing techniques
- Development of parallel ILUPACK-based solvers
  - Computation of adjacency graph G_A
  - Parallelism revealed by nested dissection on G_A
    $\rightarrow$ recursive application of vertex separator partitioning
  - Dynamic scheduling to improve load-balancing during execution
    $\rightarrow$ run-time tuned for a specific platform/application problem
  - OmpSs is a data-flow programming model
    $\rightarrow$ Exploit the data-dependencies between the tasks
    $\rightarrow$ Useful to combine different platforms in heterogenous systems

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Inverse-based IC preconditioners

- The (root-free) sparse Cholesky decomp. computes $L \in \mathbb{R}^{n,n}$, sparse unit lower triangular, and $D \in \mathbb{R}^{n,n}$, diagonal with positive entries s.t.

$$A = LDL^T$$

- In the Incomplete Cholesky (IC) factorization A is factored as

$$A = \tilde{L}\tilde{D}\tilde{L}^T + E$$

  where $E \in \mathbb{R}^{n,n}$ is a "small" perturbation matrix consisting of those entries dropped during factorization
- $M = \tilde{L}\tilde{D}\tilde{L}^T$ is applied to the original system $Ax = b$ in order to accelerate the convergence of the PCG solver

$$(\tilde{D}^{-1/2}\tilde{L}^{-1})A(\tilde{L}^{-T}\tilde{D}^{-1/2})$$

- Inverse-based IC factorizations construct $M = \tilde{L}\tilde{D}\tilde{L}^T$ s.t. $\|\tilde{L}^{-1}\| \leq \kappa$
  - $\kappa = 5$ or $\kappa = 10$ are good choices for many problems
  - Pivoting and **multilevel** methods are required in general

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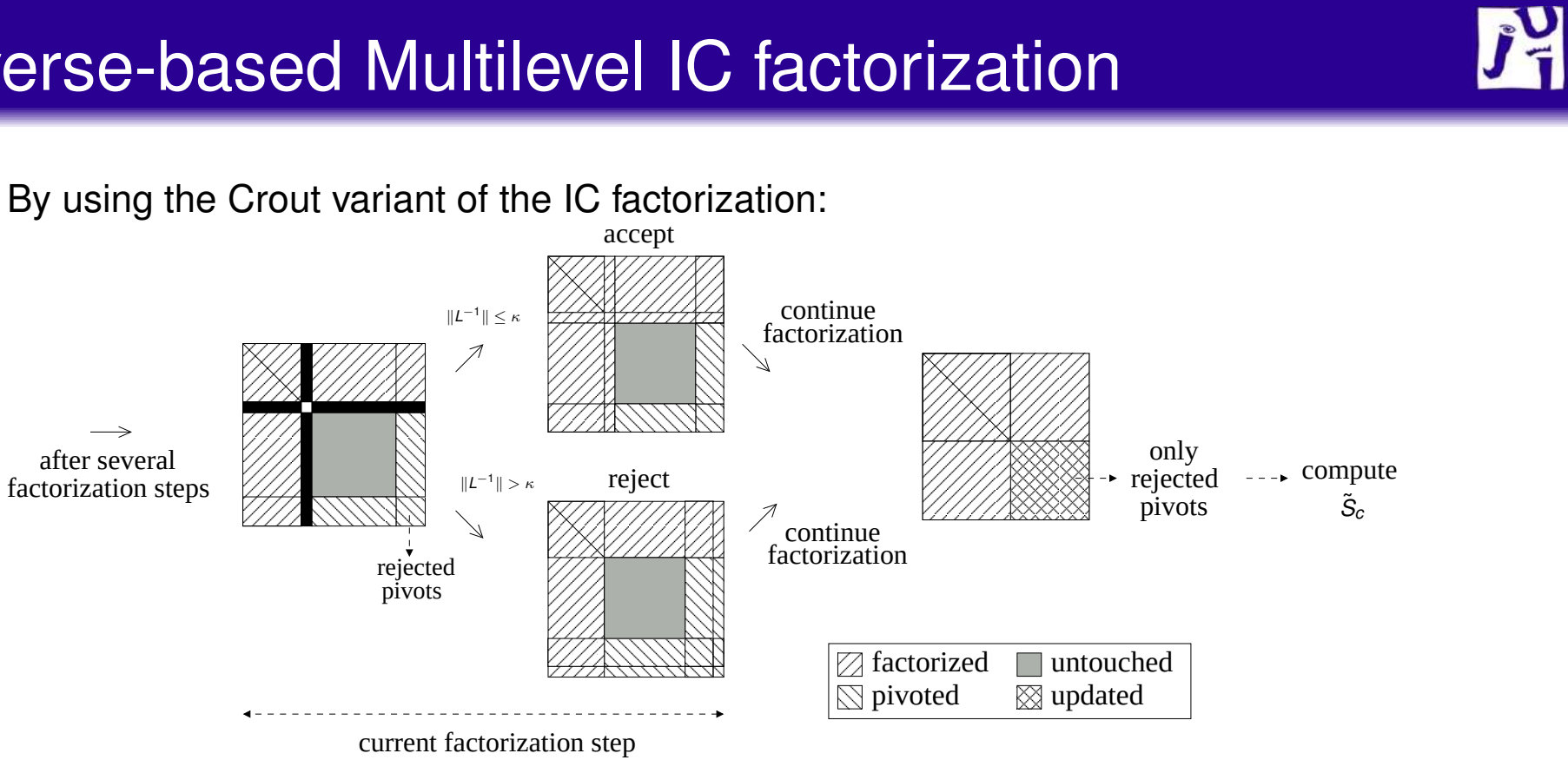
Inverse-based Multilevel IC factorization



By using the Crout variant of the IC factorization:

- The IC decomp. employs **Inverse-based** pivoting
  → Row/Col. k of $\tilde{L}$ / $\tilde{L}^T$ obtained at step k along with estimation $t_k \approx \|e_k^T \tilde{L}^{-1}\|$
  → Row/Col. s.t. $t_k > \kappa$ are moved to the bottom/right-end of the matrix
- The IC factorization **stops** when the trailing block only contains rejected pivots
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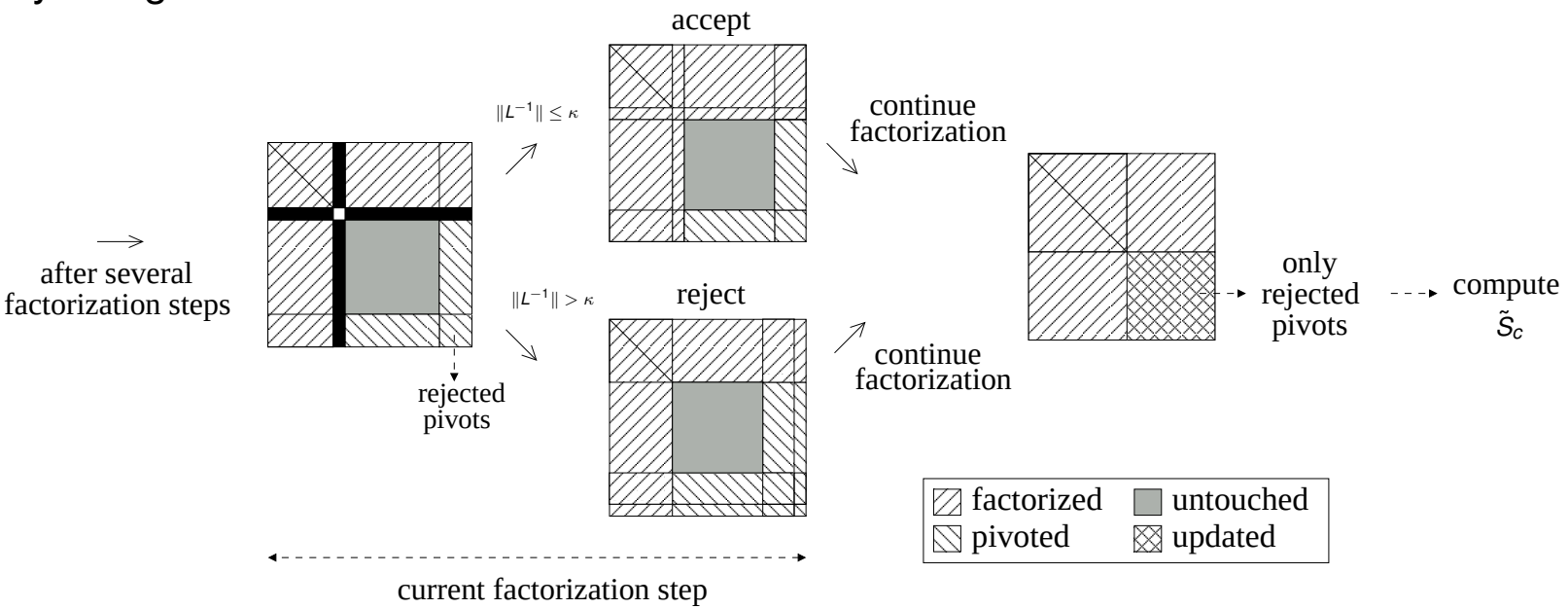

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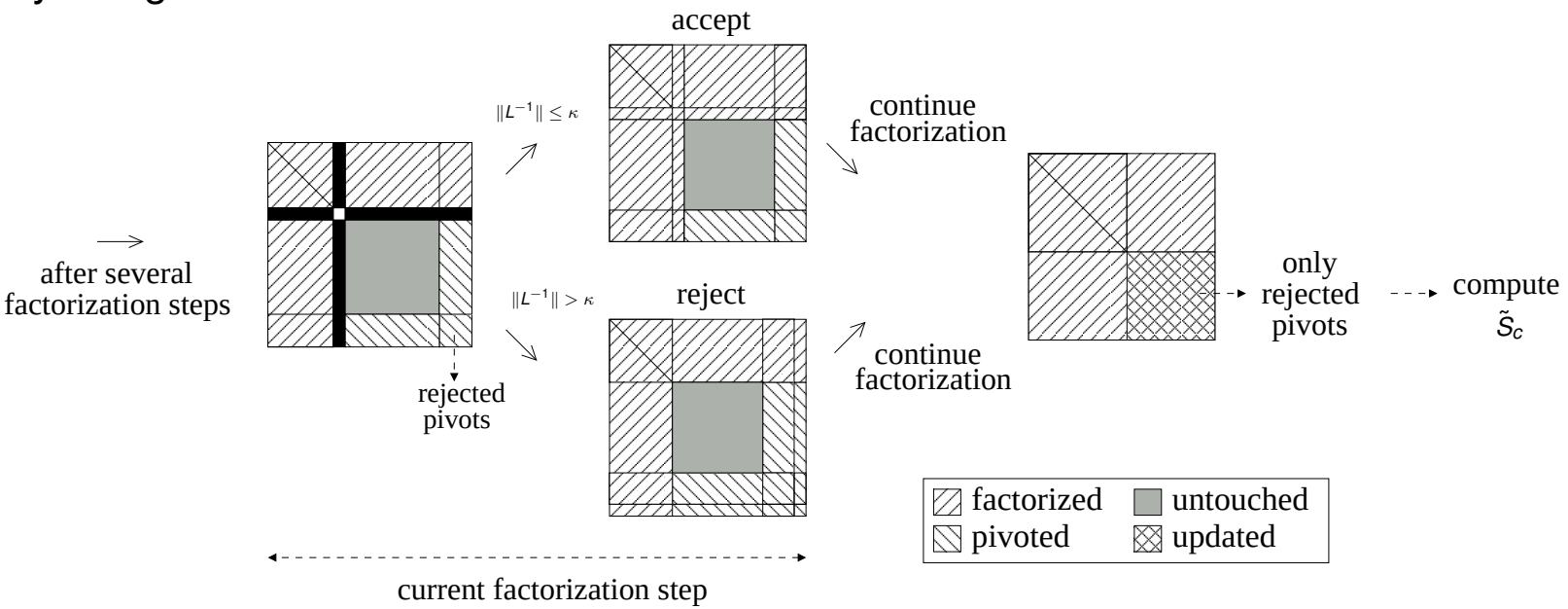

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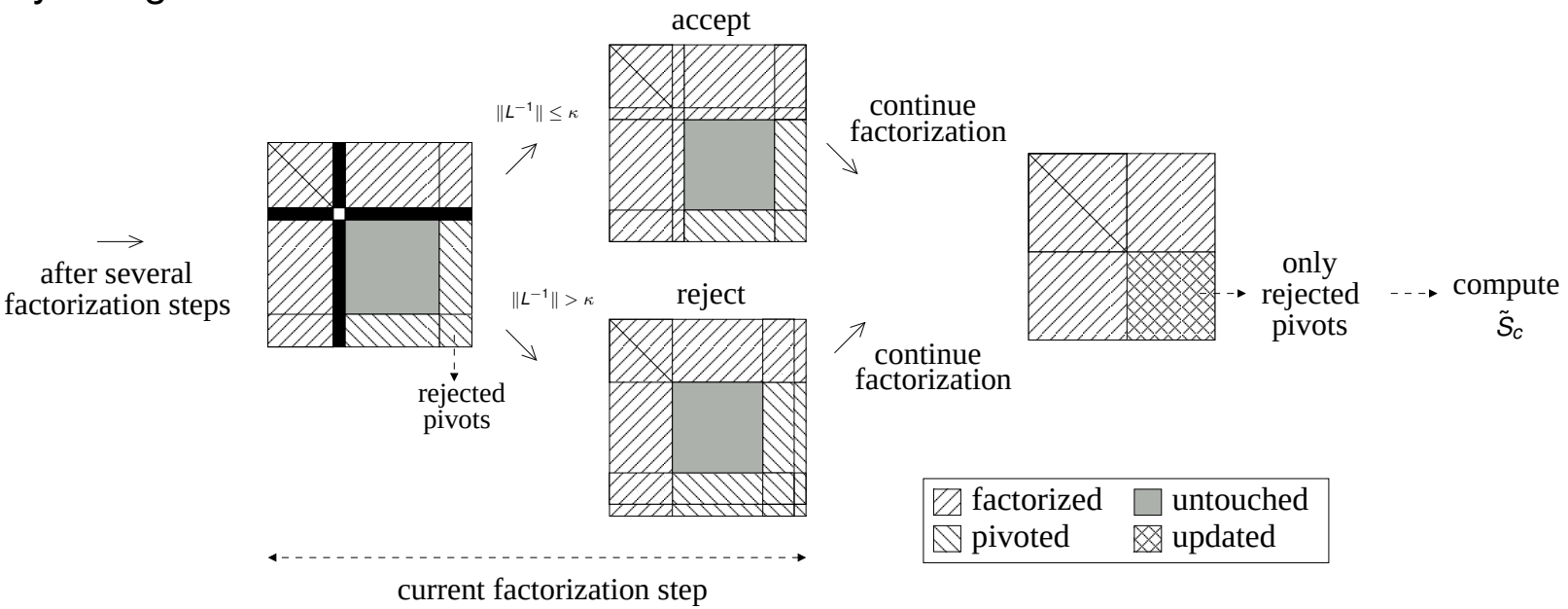


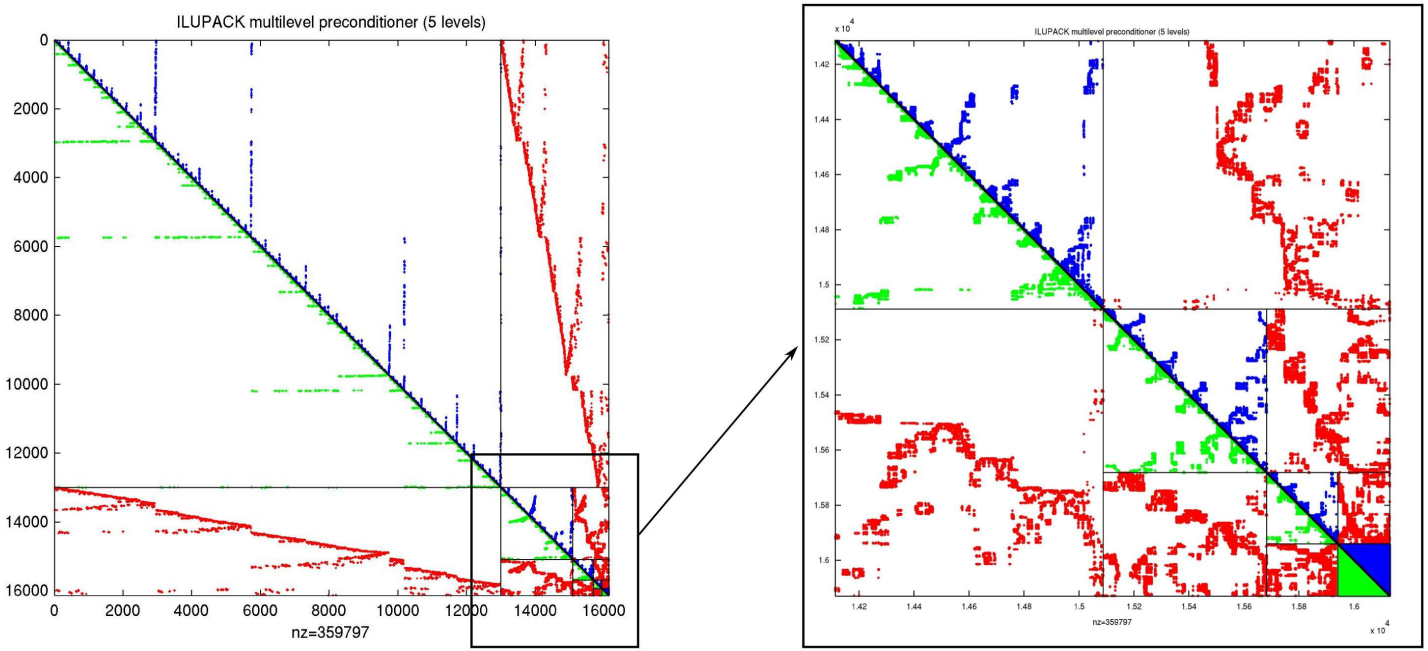
- . . . a **partial** IC decomp. of a permuted system with $\tilde{L}_B^{-1}$ satisfying $\|\tilde{L}_B^{-1}\| \lessapprox \kappa$

$$\hat{P}^T \hat{A} \hat{P} \equiv \begin{pmatrix} B & F^T \\ F & C \end{pmatrix} = \begin{pmatrix} \tilde{L}_B & 0 \\ \tilde{L}_F & I \end{pmatrix} \begin{pmatrix} \tilde{D}_B & 0 \\ 0 & \tilde{S}_c \end{pmatrix} \begin{pmatrix} \tilde{L}_B^T & \tilde{L}_F^T \\ 0 & I \end{pmatrix} + E$$

- The whole method is restarted on $\tilde{S}_c$ leading to a **multilevel** approach

Inverse-based Multilevel IC factorization

MIC factorization of five-point matrix arising from Laplace PDE discretization



$\kappa = 5$ and $\tau = 10^{-3}$ leads to a MIC factorization with 5 levels

Outline

Decomposing the problem into tasks

The key to parallelize the preconditioner computation is to decompose the IC factorization into independent tasks:

- Analyzing the sparsity pattern of A we can permute it,

$$\Pi^T A \Pi = \left(\begin{array}{cc|c} A_{11} & 0 & A_{31}^T \\ 0 & A_{22} & A_{32}^T \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right).$$

$$\left[\begin{array}{ccc|ccc|c} \hat{A}_{11} & 0 & \hat{A}_{31}^T & 0 & 0 & 0 & * \\ 0 & \hat{A}_{22} & \hat{A}_{32}^T & 0 & 0 & 0 & * \\ \hat{A}_{31} & \hat{A}_{32} & \hat{A}_{33} & 0 & 0 & 0 & * \\ \hline 0 & 0 & 0 & \bar{A}_{11} & 0 & \bar{A}_{31}^T & * \\ 0 & 0 & 0 & 0 & \bar{A}_{22} & \bar{A}_{32}^T & * \\ 0 & 0 & 0 & \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} & * \\ \hline * & * & * & * & * & * & A_{33} \end{array}\right] \rightarrow \left[\begin{array}{cccc|cc|c} A_{11} & 0 & 0 & 0 & A_{51}^T & 0 & A_{71}^T \\ 0 & A_{22} & 0 & 0 & A_{52}^T & 0 & A_{72}^T \\ 0 & 0 & A_{33} & 0 & 0 & A_{63}^T & A_{73}^T \\ 0 & 0 & 0 & A_{44} & 0 & A_{64}^T & A_{74}^T \\ \hline A_{51} & A_{52} & 0 & 0 & A_{55} & 0 & A_{75}^T \\ 0 & 0 & A_{63} & A_{64} & 0 & A_{66} & A_{76}^T \\ \hline A_{71} & A_{72} & A_{73} & A_{74} & A_{75} & A_{76} & A_{77} \end{array}\right]$$

- We can disasemble the permuted matrix into 4 submatrices

$$\left[\begin{array}{c|cc} A_{11} & A_{51}^T & A_{71}^T \\ \hline A_{51} & A_{55}^1 & A_{75}^{1\,T} \\ A_{71} & A_{75}^1 & A_{77}^1 \end{array}\right], \left[\begin{array}{c|cc} A_{22} & A_{52}^T & A_{72}^T \\ \hline A_{52} & A_{55}^2 & A_{75}^{2\,T} \\ A_{72} & A_{75}^2 & A_{77}^2 \end{array}\right], \left[\begin{array}{c|cc} A_{33} & A_{63}^T & A_{73}^T \\ \hline A_{63} & A_{66}^3 & A_{76}^{3\,T} \\ A_{73} & A_{76}^3 & A_{77}^3 \end{array}\right], \left[\begin{array}{c|cc} A_{44} & A_{64}^T & A_{74}^T \\ \hline A_{64} & A_{66}^4 & A_{76}^{4\,T} \\ A_{74} & A_{76}^4 & A_{77}^4 \end{array}\right]$$

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Decomposing the problem into tasks

The key to parallelize the preconditioner computation is to decompose the IC factorization into independent tasks:

- In parallel, we can compute the partial IC factorization of each submatrix

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- A partial IC factorization of matrix A is computed:

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The key to parallelize the preconditioner computation is to decompose the IC factorization into independent tasks:

- In parallel, we can compute the partial IC factorization of each submatrix

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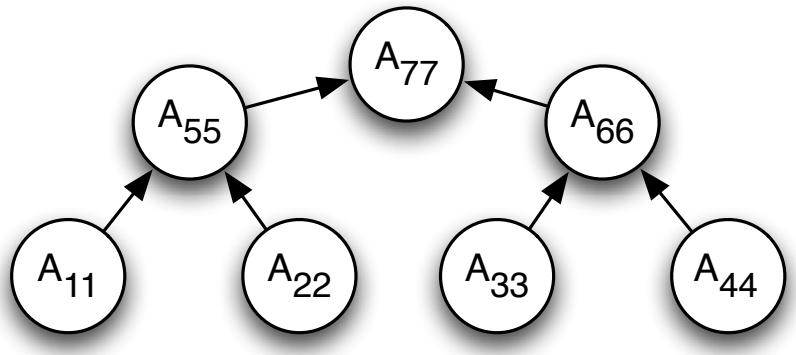

- Each node refers to a block to be factorized, while the edges define the data dependency.
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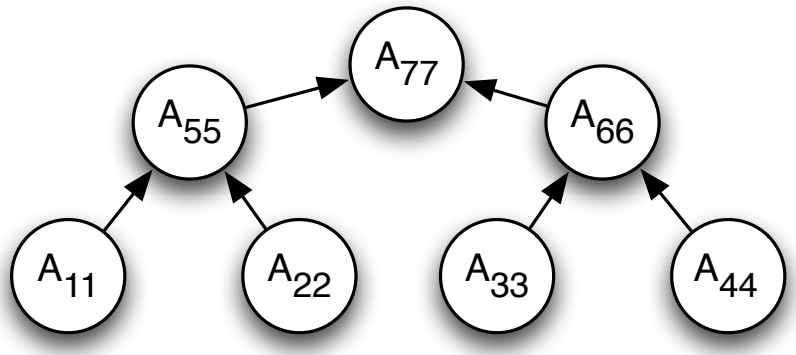

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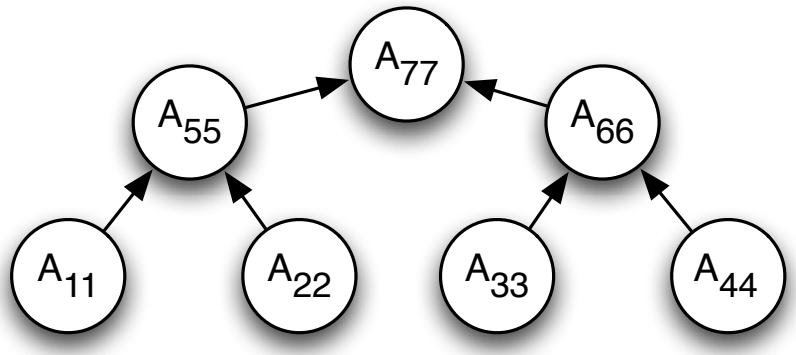

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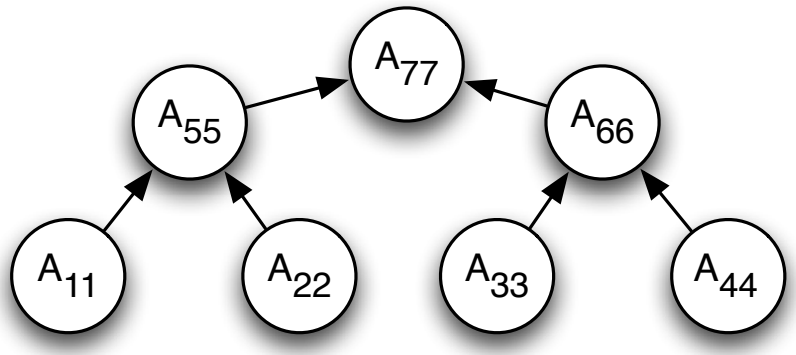

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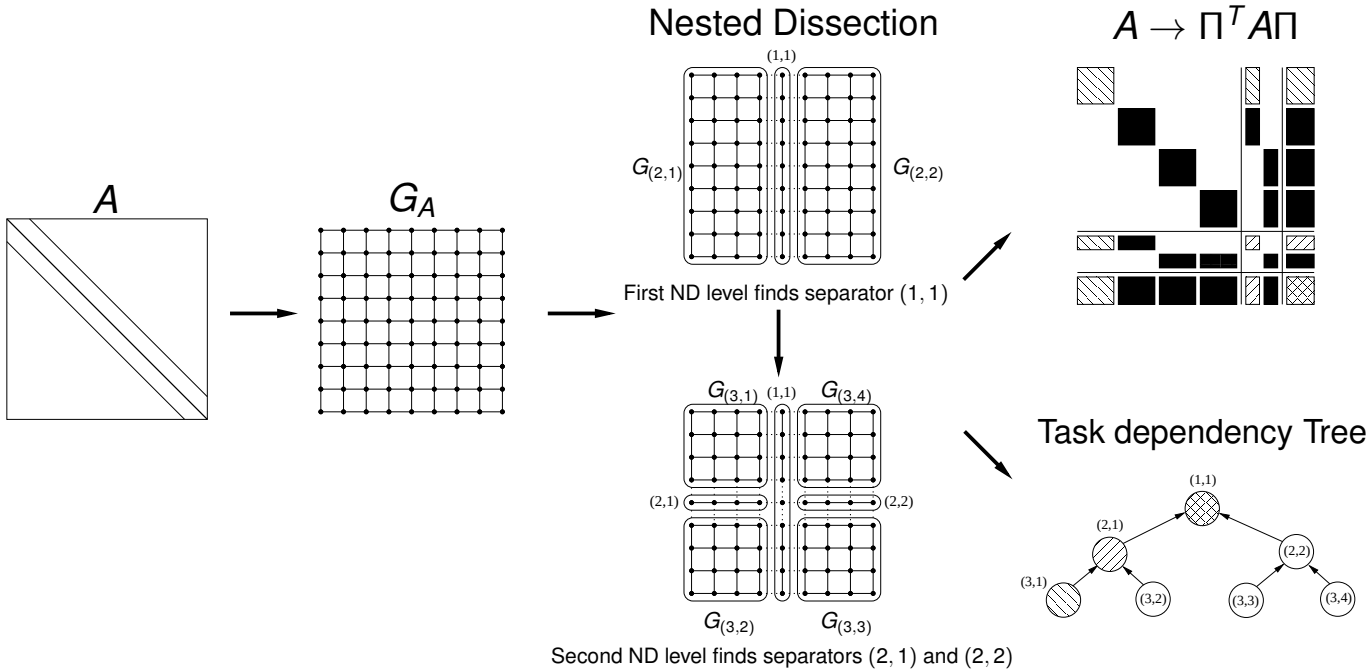

- Task-level parallelism revealed by nested dissection on adjacency graph G_A
  $\rightarrow$ fast and efficient **multilevel** (MLND) variants in METIS and SCOTCH
- Parallelization driven by the dependencies "captured" in the task tree
  $\rightarrow$ subdomains **first** eliminated independently, **then** separators $(2, 1)$ and $(2, 2)$ in parallel, **finally** the root separator $(3, 1)$

Parallel Inverse-based Multilevel IC factorization

The parallelization of ILUPACK must consider two multilevel procedures

- ILUPACK computes Multilevel IC factorization
  $\rightarrow$ Numerical levels related to inverse-based IC factorizations
- Parallelization of IC factorization
  $\rightarrow$ Structural levels computed by nested disection algorithm

The solution is to combine both multilevel procedures

- Structural levels are initially used to reveal the concurrence
- Numerical levels appears during the MIC factorization of each submatrix

The Multilevel IC factorization must be adjusted when it is applied on a submatrix

- Where to move the rejected pivots
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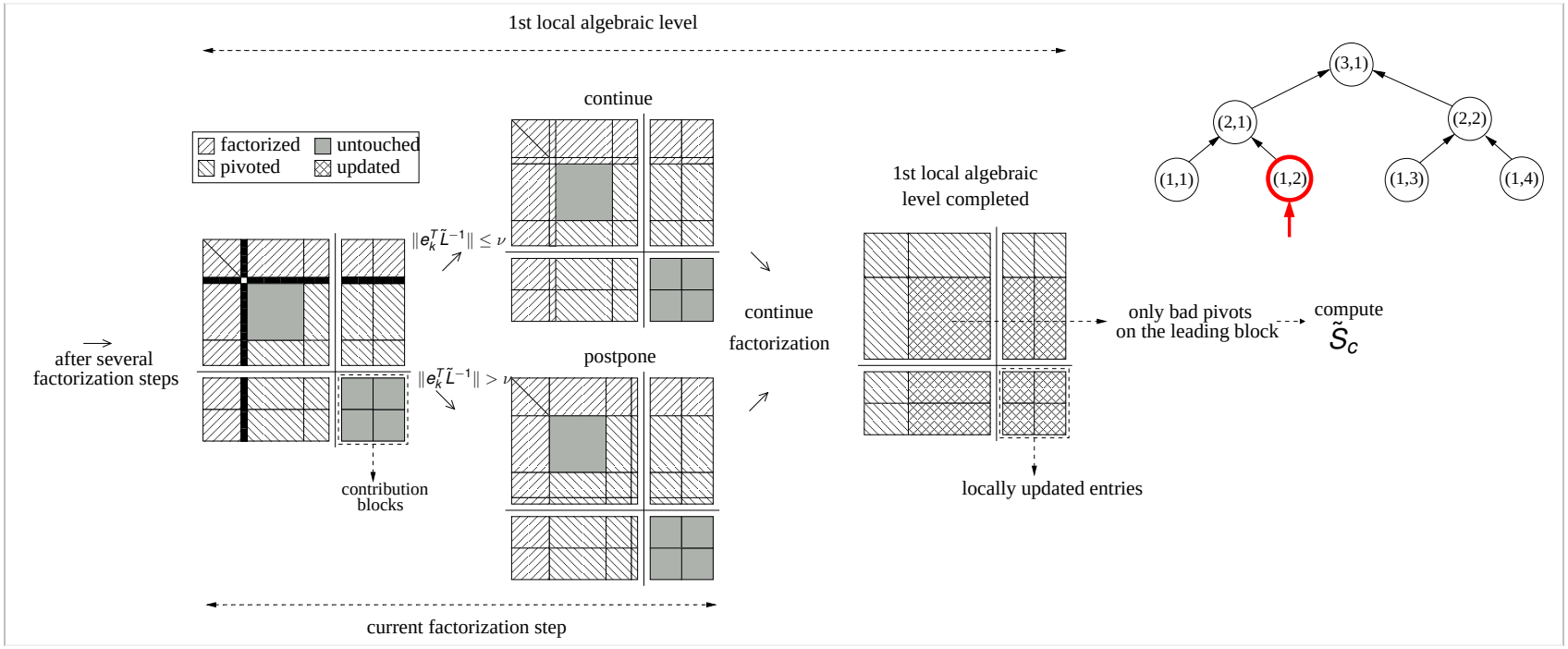
The tasks compute a local MIC **restricted to the leading block** of the submatrix



- IC decomp. employs **restricted** Inverse-based pivoting
  → Rejected pivots **are only moved** to the bottom/right-end of the leading block
- IC decomp. stops if the trailing of the **leading block** only contains rejected pivots
  → Rejected pivots still “resolved” inside the task by entering next MIC level

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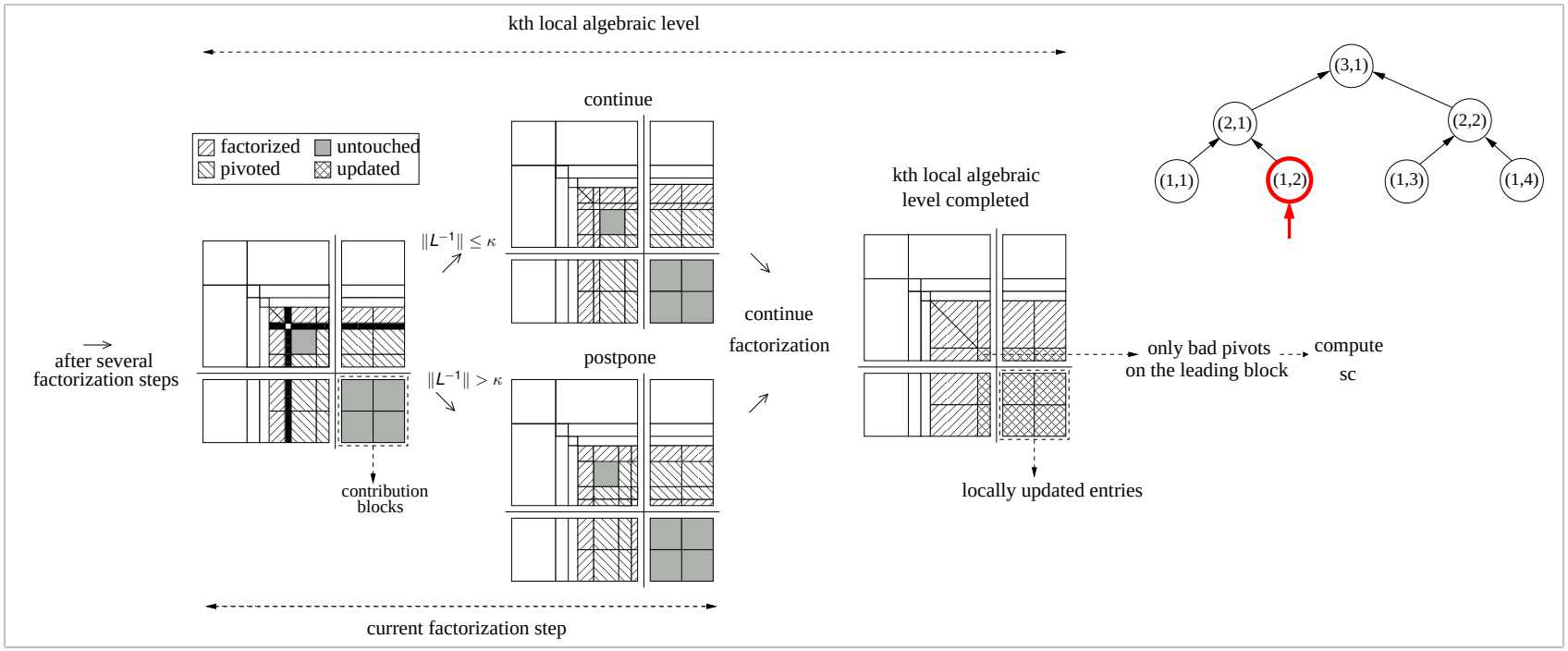
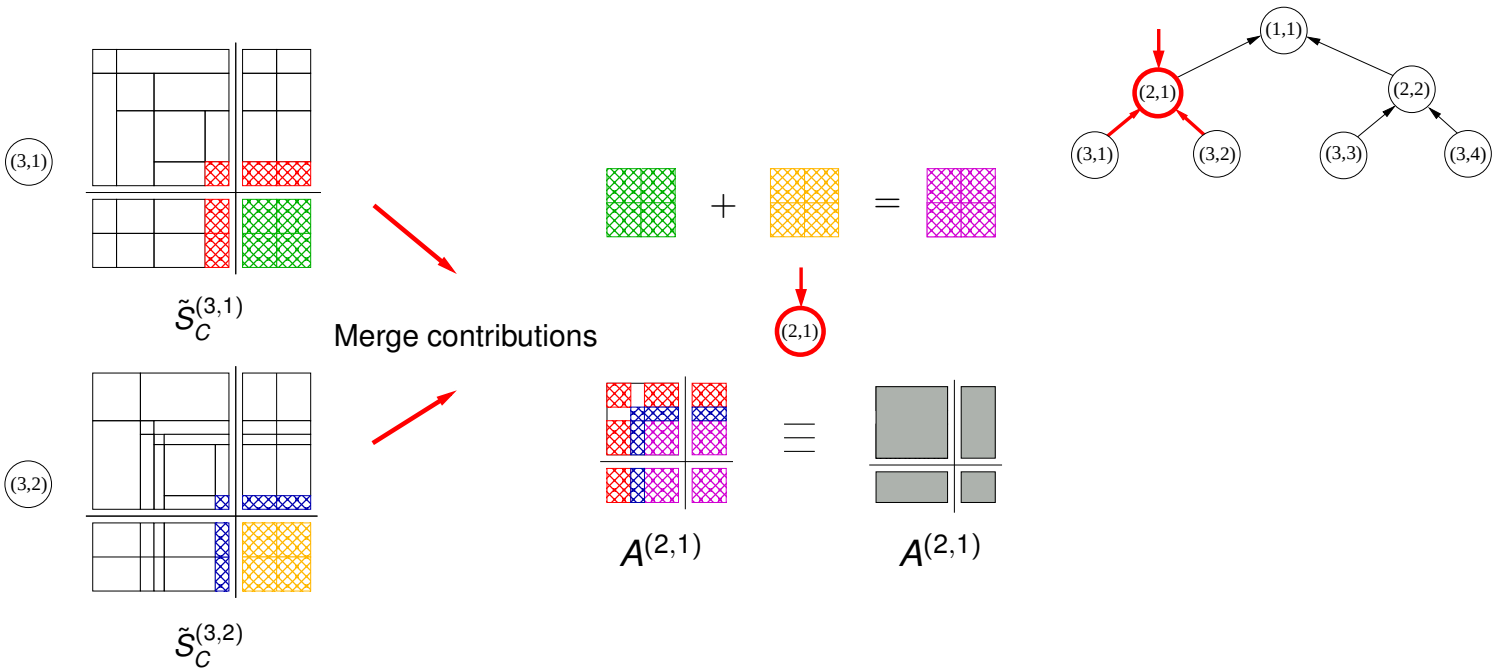
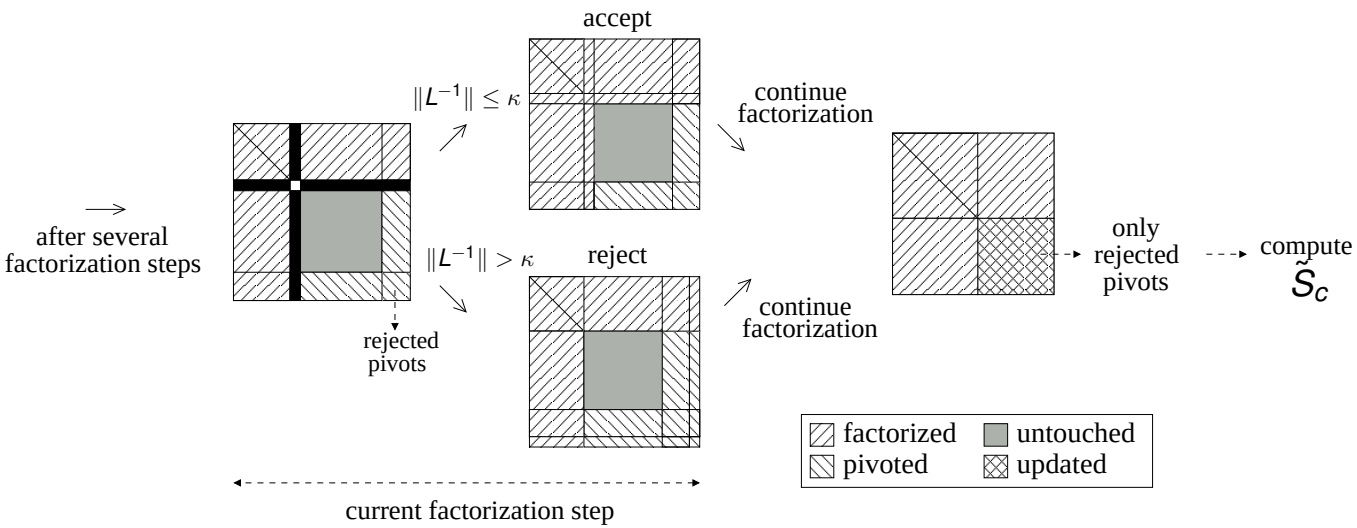


- When the set of rejected pivots is “small enough” the local MIC **stops**
  → The task “sends” the local contributions to the parent task

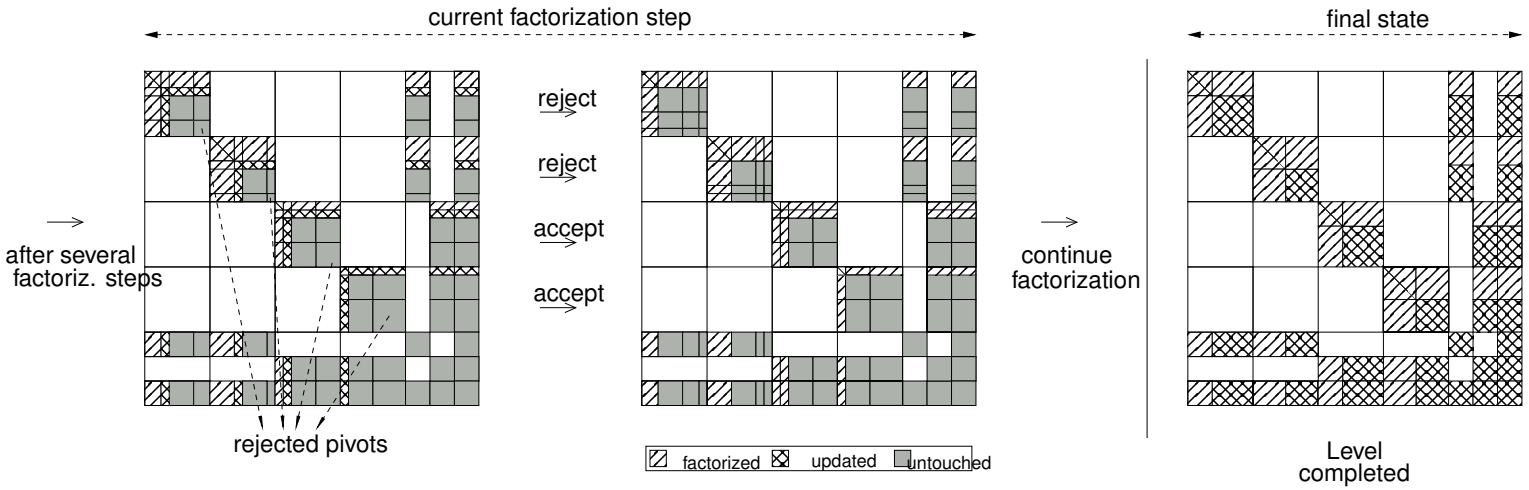
Parallel inverse-based MIC factorization



- The parent task merges contributions resulting from its children
- A new submatrix is constructed for the local MIC decomp. of the separator task
  - $\rightarrow$ Bad pivots resulting from its children copied on top of the leading block
  - $\rightarrow$ Contributions blocks added



inverse-based Multilevel IC factorization

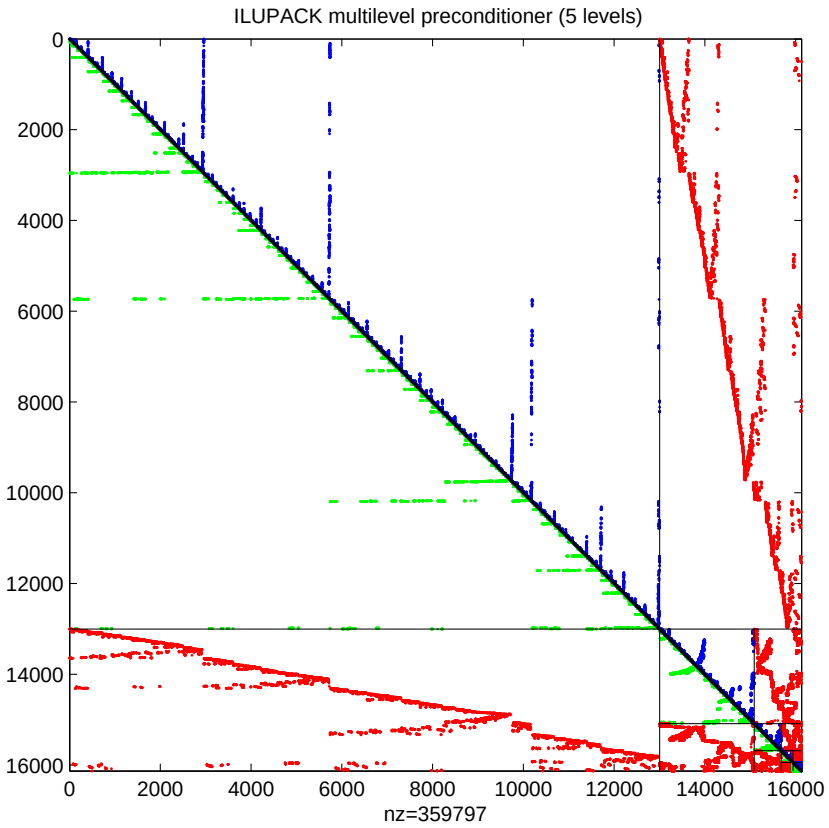


Parallel inverse-based Multilevel IC factorization

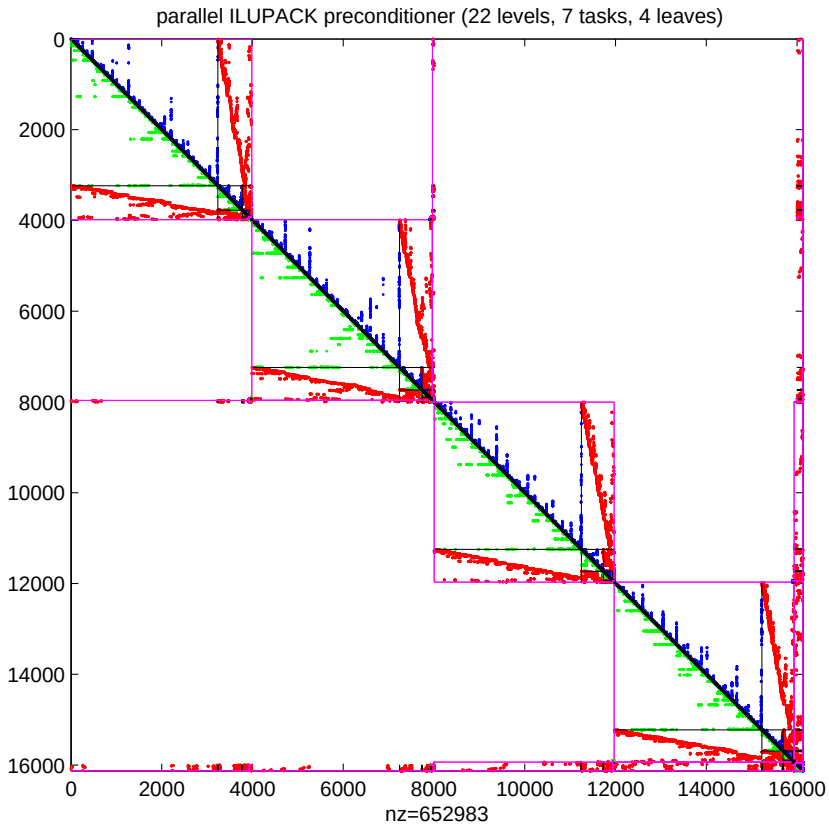
Parallel vs Sequential Inverse-based MIC factoriz.

Factorization of five-point matrix arising from Laplace PDE discretization

MIC (5 levels)



Parallel MIC (4 leaves, 22 levels)



$\kappa = 5$ and $\tau = 10^{-3}$

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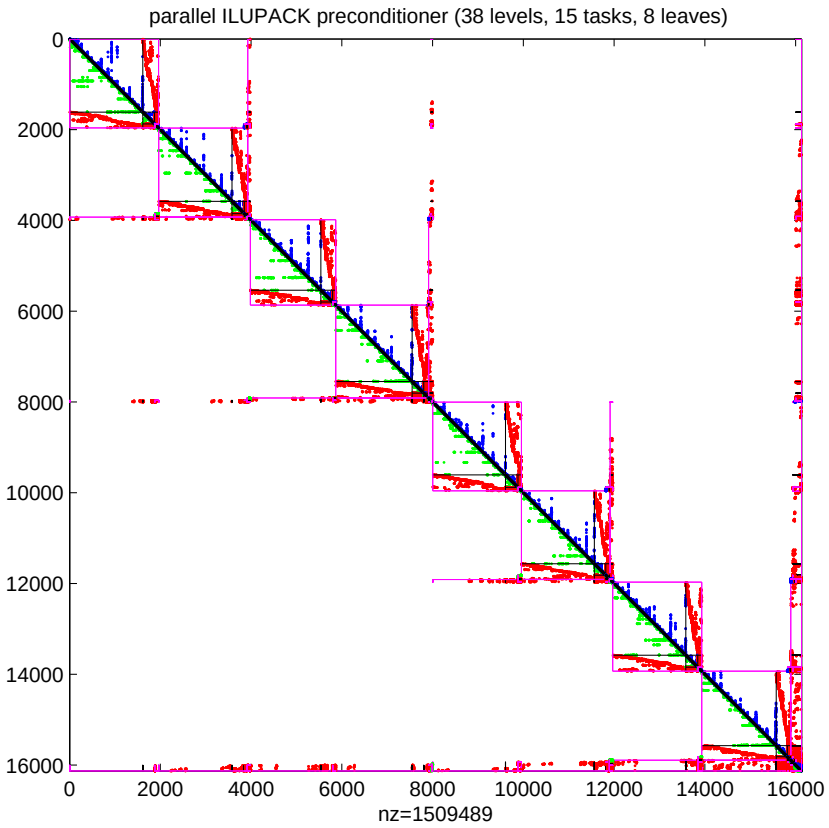
Parallel MIC (8 leaves, 38 levels)



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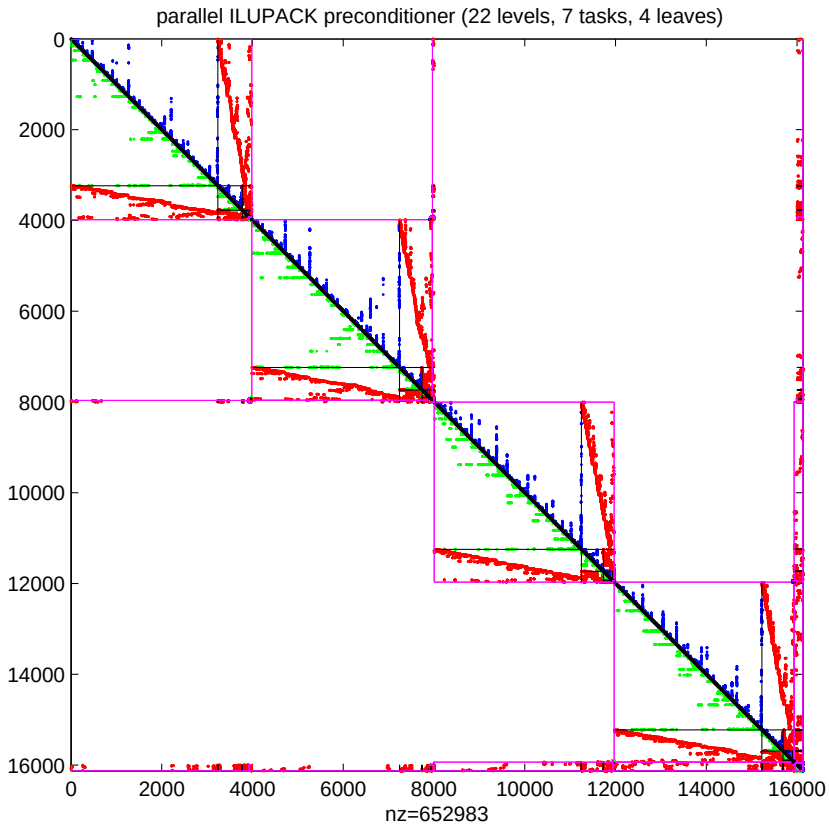

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OmpSs programming model

On the way to Exascale, one of the main challenges is the parallel programming revolution:

- In the classical programming models, the code defines
  - Where to place data
  - What to run and where
  - How to communicate
- In the new programming models, the questions have changed,
  - What do I need to compute?
  - What data do I need to use?
  - Hints (not necessarily very precise) on potential concurrency, locality, ...
- For the programmers, the main feature of the new programming models is to write a single source useful for any target (multicores, multicomputers, heterogeneous systems with GPU?s, ...)

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OmpSs is a new programming model, which extends OpenMP:

- Define a single address space
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- Tasks annotated with directionality clauses,

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From the point of view of the programmer a single naming space exists

- The standard sequential/shared memory address space

From the point of view of the runtime and target platform, different possible scenarios

- Pure SMP:
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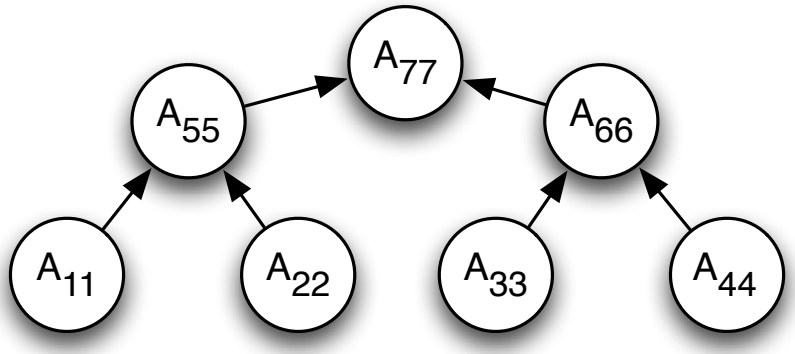
Preconditioner computation on OmpSs

Task dependency tree:



We define a structure called dag[][] to define the dependencies in OmpSs

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Task id. j		0	1	2	3	4	5	6
Left descendant	dag[0][j]	–	–	–	–	0	2	4
Right descendant	dag[1][j]	–	–	–	–	1	3	5
Ancestor	dag[2][j]	4	4	5	5	6	6	–

The dag is created and initialized before the parallel computation begins

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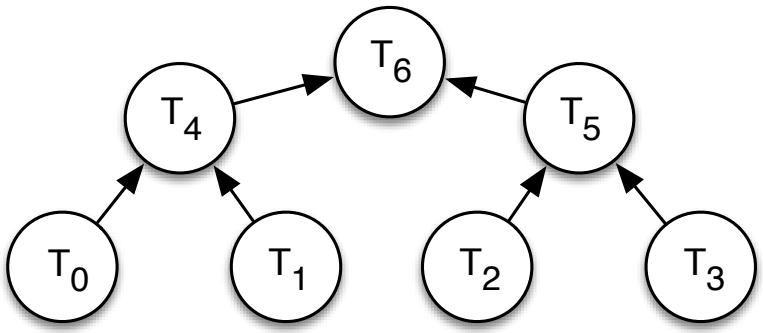

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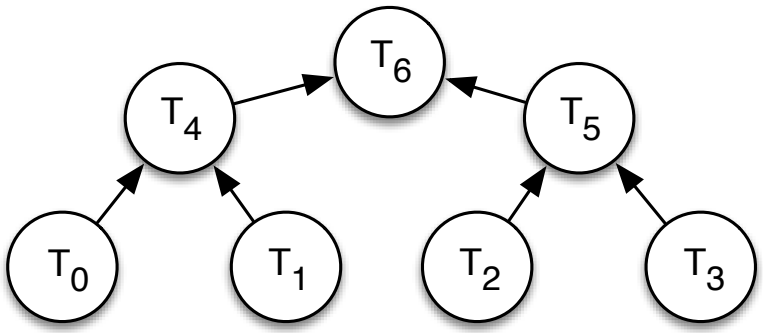
Ancestor	`dag[2][`j`]`	4	4	5	5	6	6	–

The dag is created and initialized before the parallel computation begins

Preconditioner computation on OmpSs

Task dependency tree:



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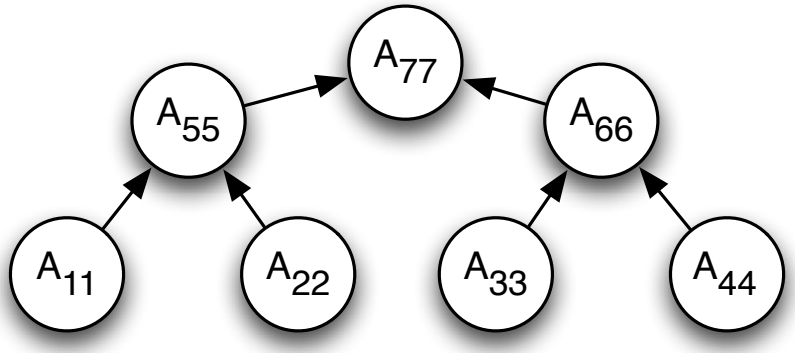

Given the complexity of the data structure, we use a vector of sentinels to define the data dependency.

```
for(i=0; i<dimL; i++){
  tsk_id=i;
  left=dag[0][tsk_id]; right=dag[1][tsk_id];
  #pragma omp task in(v_sentinel[left],v_sentinel[right]) \
                   out(v_sentinel[tsk_id])
  ILUPrecond(spMat, spFact, taskid, v_sentinel, dag);
}
```

Preconditioner computation on OmpSs

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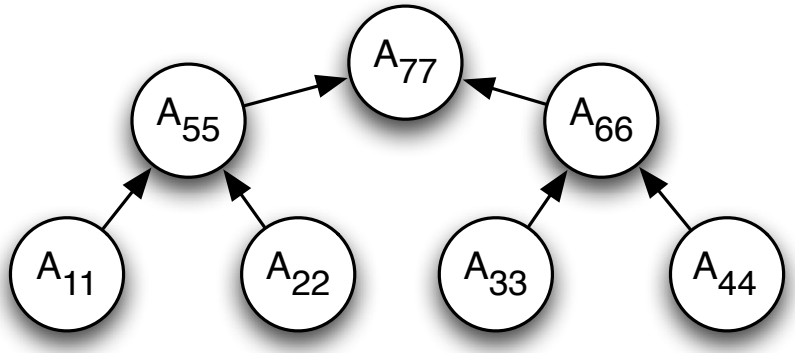

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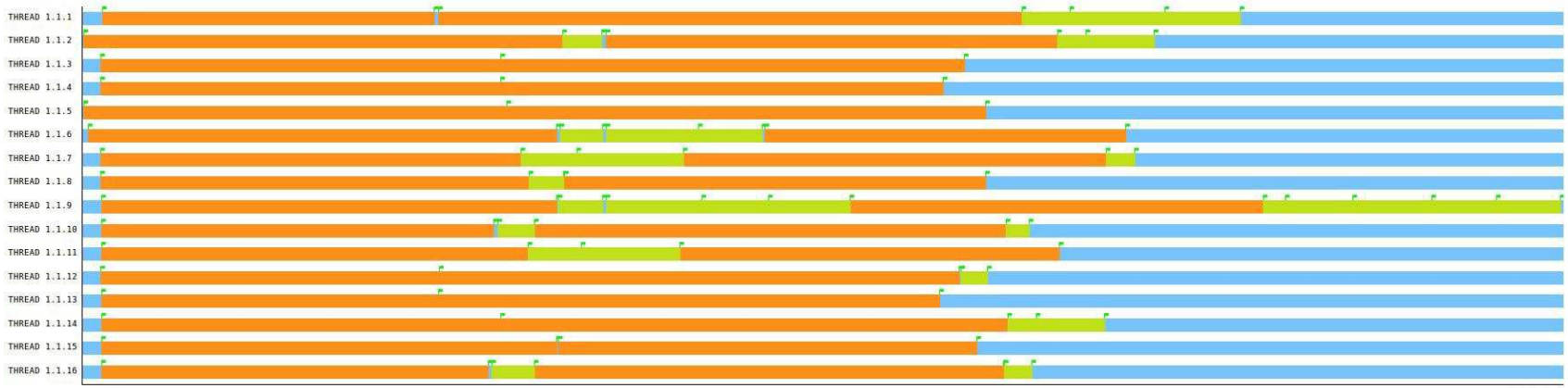
```

Preconditioner computation on OmpSs

PARAVER trace of the preconditioner computation (32 leaves & 16 threads):



Analyzing the trace, we observe that:

- The leaf tasks (orange) are larger than the non-leaf tasks (green)
- Some small tasks are processed before the larger tasks

Therefore, some priorities could be defined to improve performances:

- To process leaf tasks before non-leaf tasks
- ...

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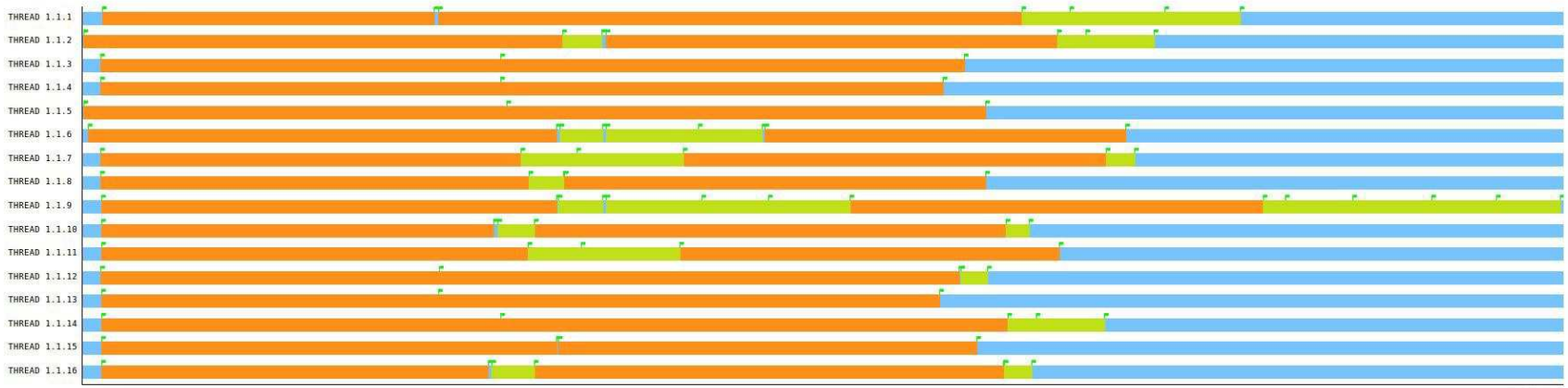

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Preconditioner computation on OmpSs

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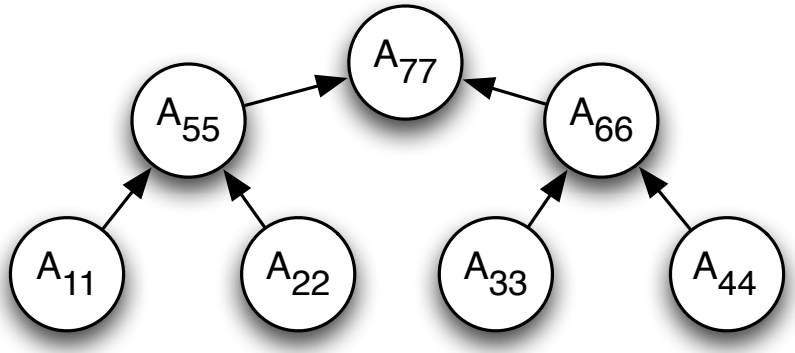
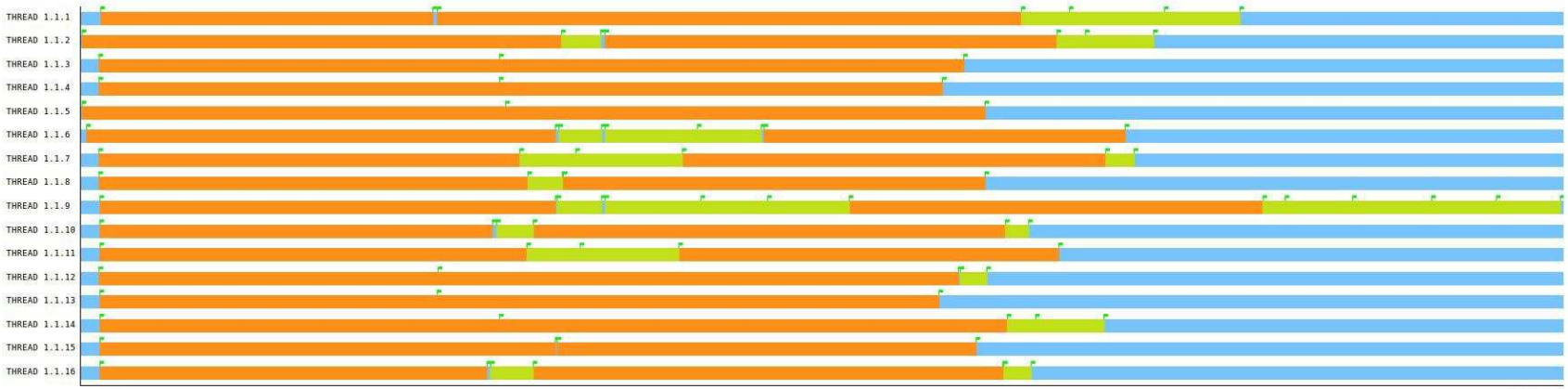


The priority could be used to improve performances

```
for(i=0; i<dimL; i++){
  tsk_id=tsk_ord[i]; prio=tsk_prio[i];
  left=dag[0][tsk_id]; right=dag[1][tsk_id];
  #pragma omp task in(v_sentinel[left],v_sentinel[right]) \
                   out(v_sentinel[tsk_id]) priority(prio)
  ILUPrecond(spMat, spFact, tsk_id, v_sentinel, dag);
}
```

Preconditioner computation on OmpSs

PARAVER trace without priority:



PARAVER trace with priority:

Outline

Experimental Framework

- Target platforms:
  - 4 AMD Opteron 6276 Processors (16 cores, total of 64):
    - 2.10 GHz
    - 64 Gbytes RAM
  - 2 Intel Xeon E5-2670 (8 cores, total of 16)
    - 2.60 GHz
    - 32 Gbytes RAM
  - All executions use up to 16 cores.
- IEEE double precision
- OmpSs version: Nanox 0.7a, mcxx 1.99

Example Benchmark

We consider the Laplacian PDE equation

$$-\Delta u = f$$

in a $3D$ unit cube $\Omega = [0,1]^3$ with Dirichlet boundary conditions $u = g$ on $\partial\Omega$

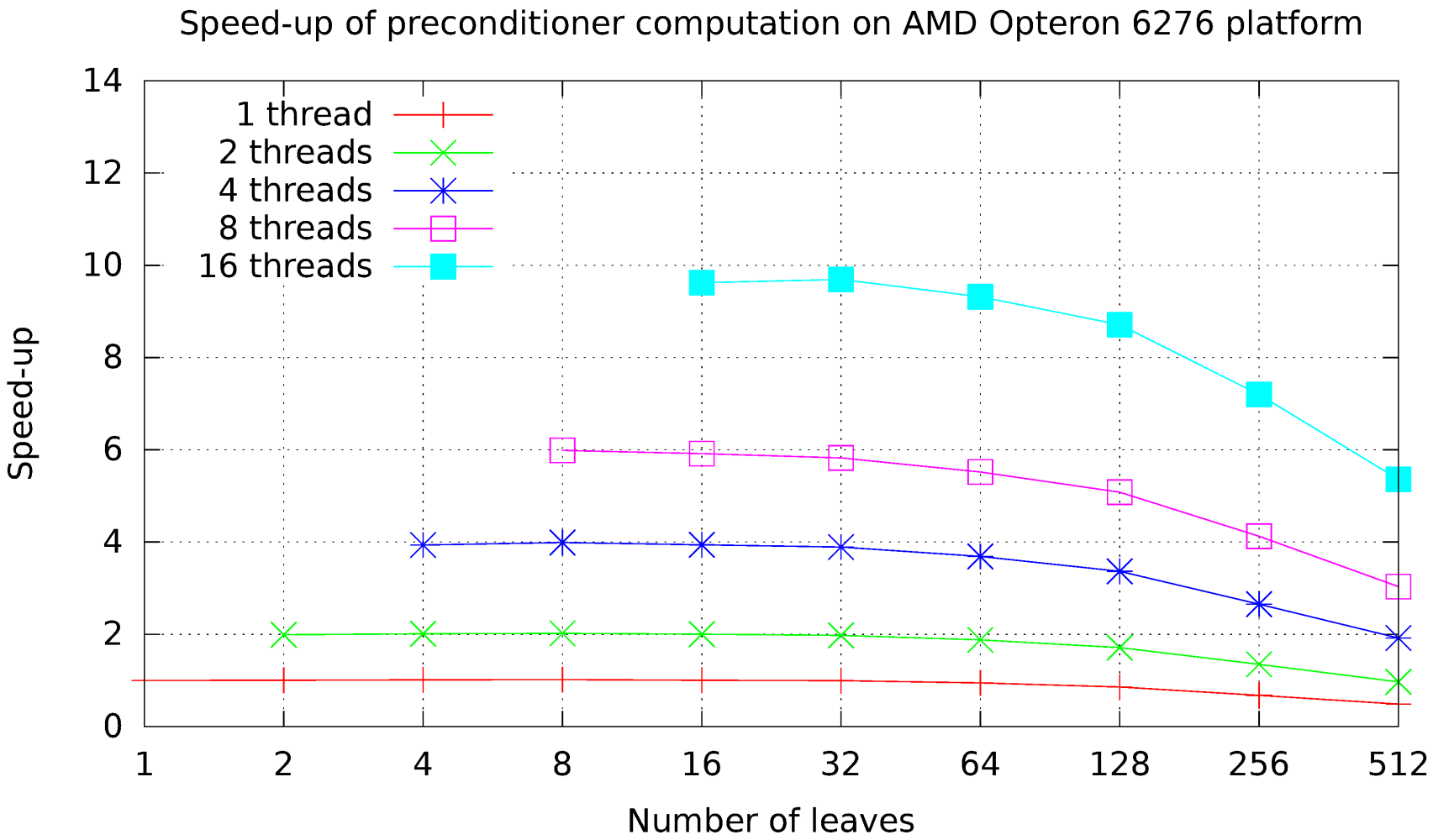
For the discretization . . .

- Ω replaced by a $N_x \times N_y \times N_z$ uniform grid
- Δu approximated by centered finite differences
  $\rightarrow$ seven-point-star difference stencil

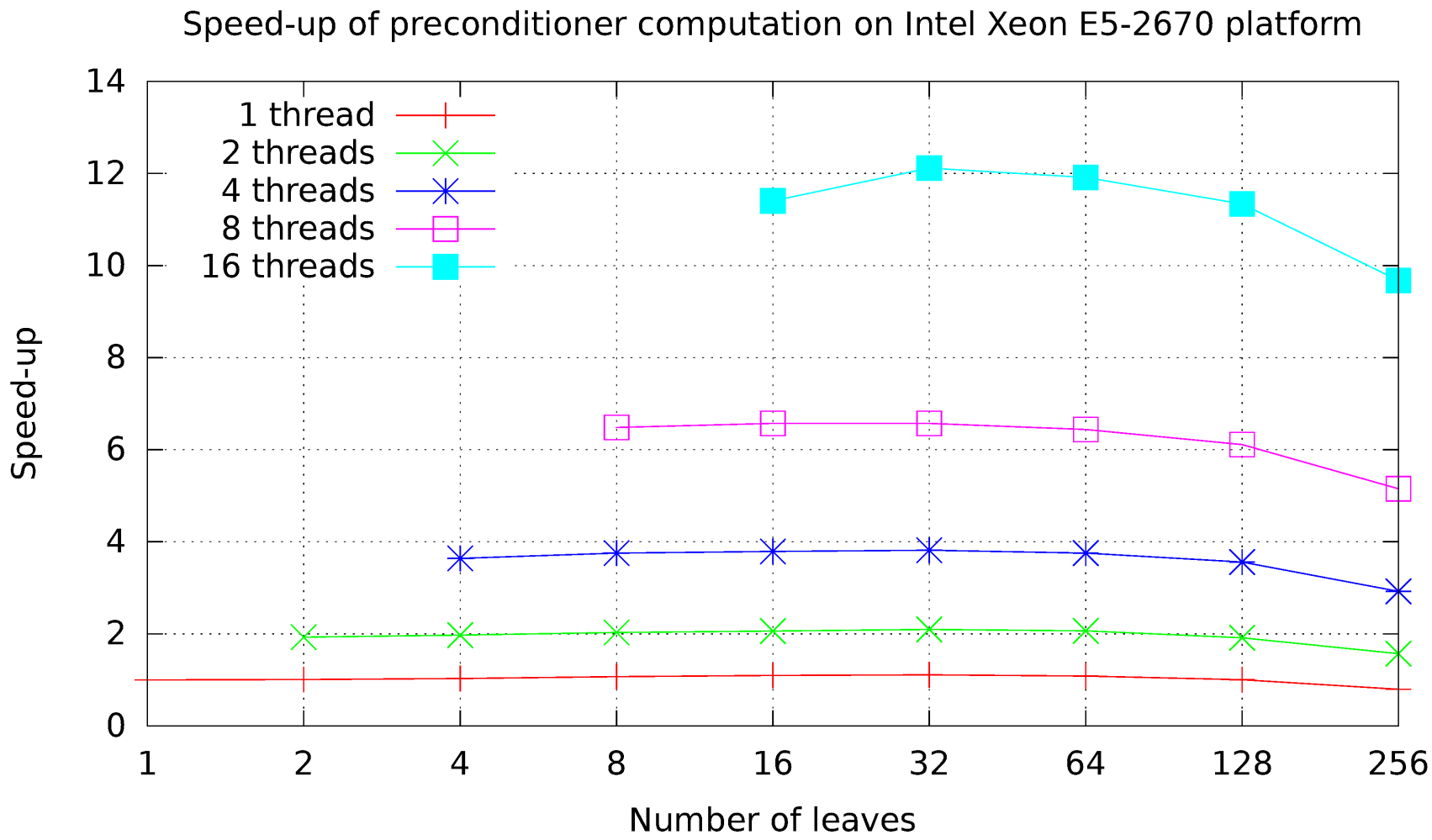
In this talk, we only consider the largest benchmark problem that fitted into the main memory of the target machines

$N_x \times N_y \times N_z$	n	nnz_A/n
$200 \times 200 \times 200$	8,000,000	7

OMPSS preconditioner speed-up in AMD



OMPSS preconditioner speed-up in INTEL



Outline

Preconditioned Conjugate Gradient

Initialize $k, x_0, r_0, z_0, d_0, \beta_0, \tau_0$
while $((k < k_{max})\&\&(\tau_k > \tau_{min}))$
$w_k := Ad_k$
$\gamma_k := d_k^T w_k$
$\rho_k := \beta_k / \gamma_k$
$x_{k+1} := x_k + \rho_k d_k$
$r_{k+1} := r_k - \rho_k w_k$
$z_{k+1} := M^{-1} r_{k+1}$
$\alpha_k := \beta_k$
$\beta_{k+1} := r_{k+1}^T z_{k+1}$
$\alpha_k := \beta_{k+1} / \alpha_k$
$d_{k+1} := z_{k+1} + \alpha_k d_k$
$\tau_{k+1} := ||r_{k+1}||_2$
$k := k + 1$
end while

Preconditioned Conjugate Gradient

Initialize $k, x_0, r_0, z_0, d_0, \beta_0, \tau_0$	
while $((k < k_{max})\&\&(\tau_k > \tau_{min}))$	
$w_k := Ad_k$	SPMV
$\gamma_k := d_k^T w_k$	DOT
$\rho_k := \beta_k / \gamma_k$	
$x_{k+1} := x_k + \rho_k d_k$	AXPY
$r_{k+1} := r_k - \rho_k w_k$	AXPY
$z_{k+1} := M^{-1} r_{k+1}$	Preconditioning
$\alpha_k := \beta_k$	
$\beta_{k+1} := r_{k+1}^T z_{k+1}$	DOT
$\alpha_k := \beta_{k+1} / \alpha_k$	
$d_{k+1} := z_{k+1} + \alpha_k d_k$	AXPY-like
$\tau_{k+1} := \|r_{k+1}\|_2$	2-norm
$k := k + 1$	
end while	

Preconditioned Conjugate Gradient

Initialize $k, x_0, r_0, z_0, d_0, \beta_0, \tau_0$
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$\tau_{k+1} := ||r_{k+1}||_2$
$k := k + 1$
end while

SPMV $\rightarrow$ on leaves
DOT $\rightarrow$ on leaves + reduction

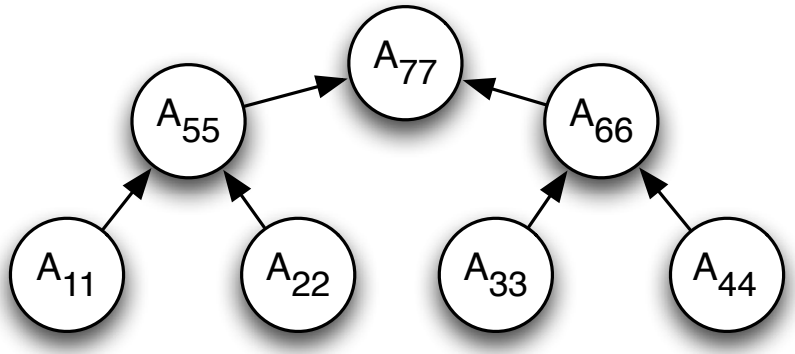
AXPY $\rightarrow$ on leaves
AXPY $\rightarrow$ on leaves
Preconditioning $\rightarrow$ cross. up + down

DOT $\rightarrow$ on leaves + reduction

AXPY-like $\rightarrow$ on leaves
2-norm $\rightarrow$ on leaves + reduction

Preconditioned Conjugate Gradient on OmpSs

Task dependency tree:



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Block		A_{11}	A_{22}	A_{33}	A_{44}	A_{55}	A_{66}	A_{77}
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Left descendant	`dag[0][`j`]`	0	1	2	3	0	2	4
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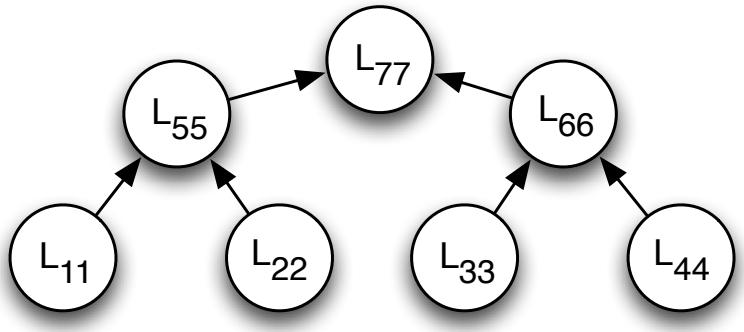
Preconditioned Conjugate Gradient on OmpSs

Task dependency tree:

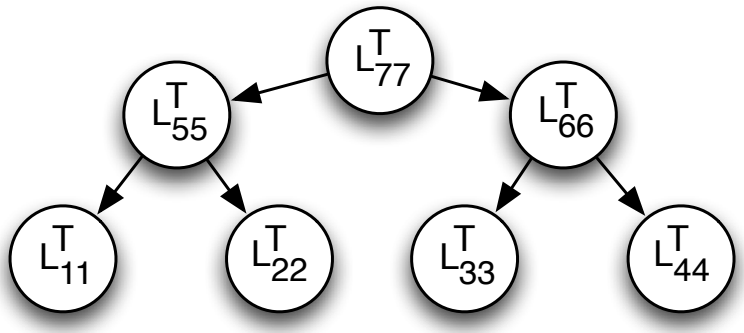


In the forward resolution, the tree is crossed from bottom to top:

```
for(i=0; i<dimL; i++){
  tsk_id=tsk_ord[i]; prio=tsk_prio[i];
  left=dag[0][tsk_id]; right=dag[1][tsk_id];
  #pragma omp task in(v_sentinel[left],v_sentinel[right]) \
                   out(v_sentinel[tsk_id]) priority(prio)
  ILUSolveUp(spFact, r, y, tsk_id,  v_sentinel, dag);
}
```

Preconditioned Conjugate Gradient on OmpSs

Task dependency tree:

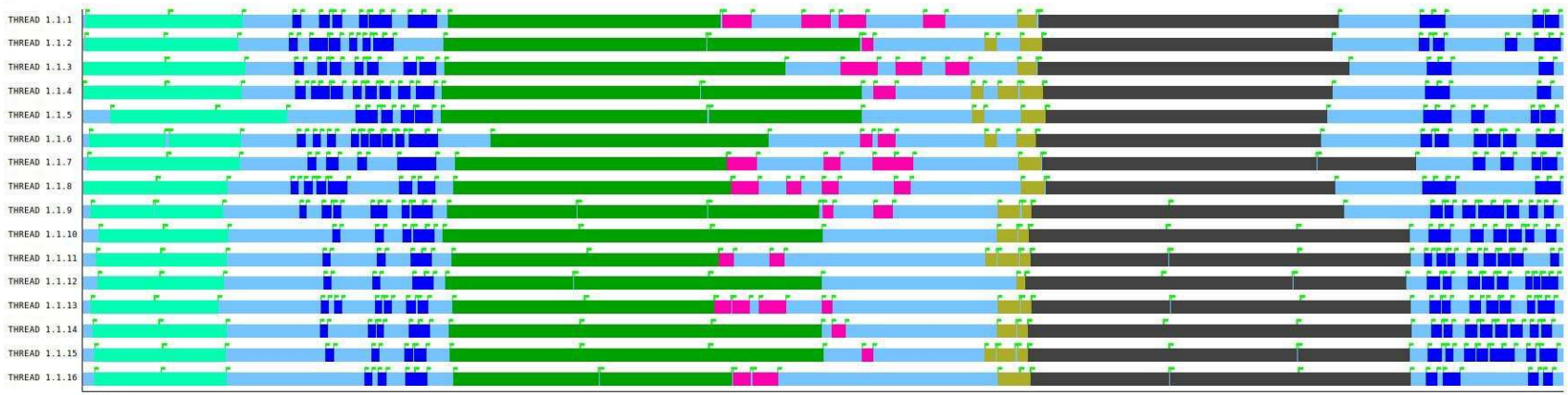


In the backward resolution, the tree is crossed from top to bottom:

```
for(i=0; i<dimL; i++){
  tsk_id=tsk_ord[i]; prio=tsk_prio[i];
  father=dag[2][tsk_id];
  #pragma omp task in(v_sentinel[father]) \
                   out(v_sentinel[tsk_id]) priority(prio)
  ILUSolveDown(spFact, y, z, tsk_id, v_sentinel, dag);
}
```

Preconditioned Conjugate Gradient on OmpSs

PARAVER trace of one iteration (32 leaves & 16 threads): many small tasks



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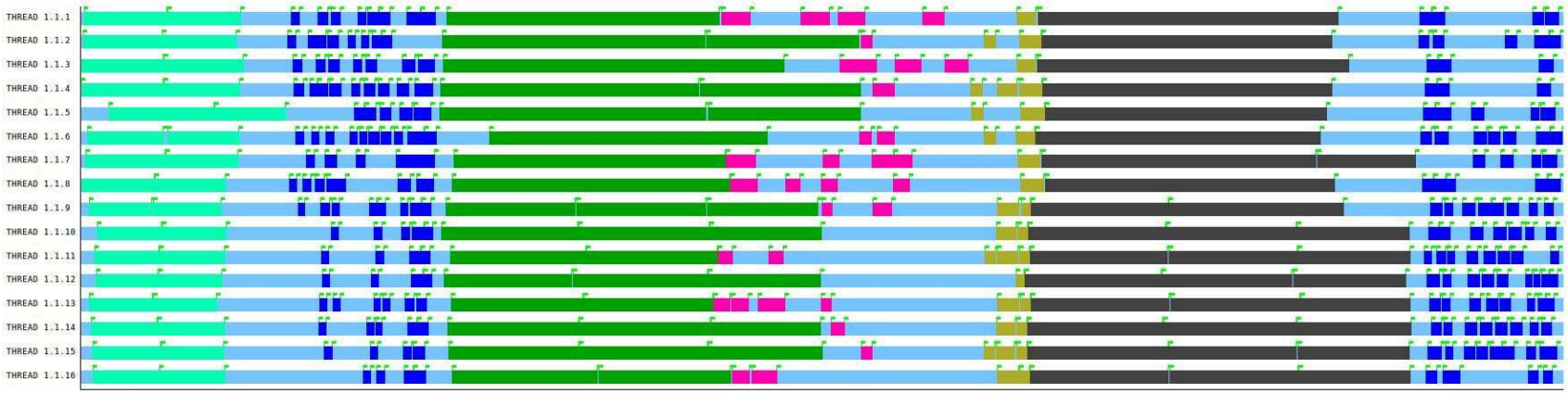
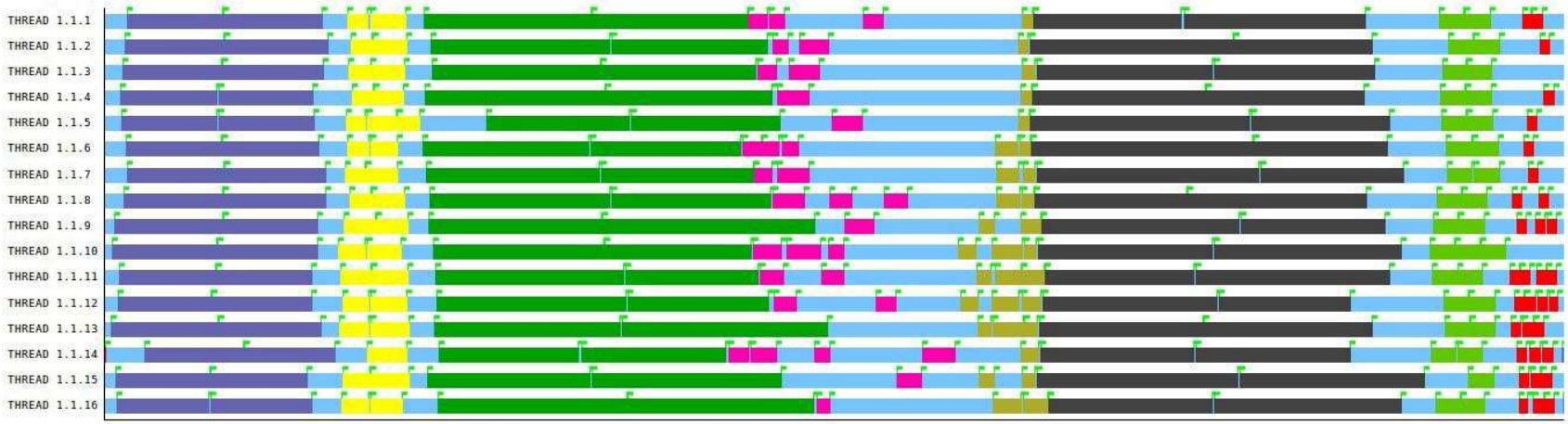
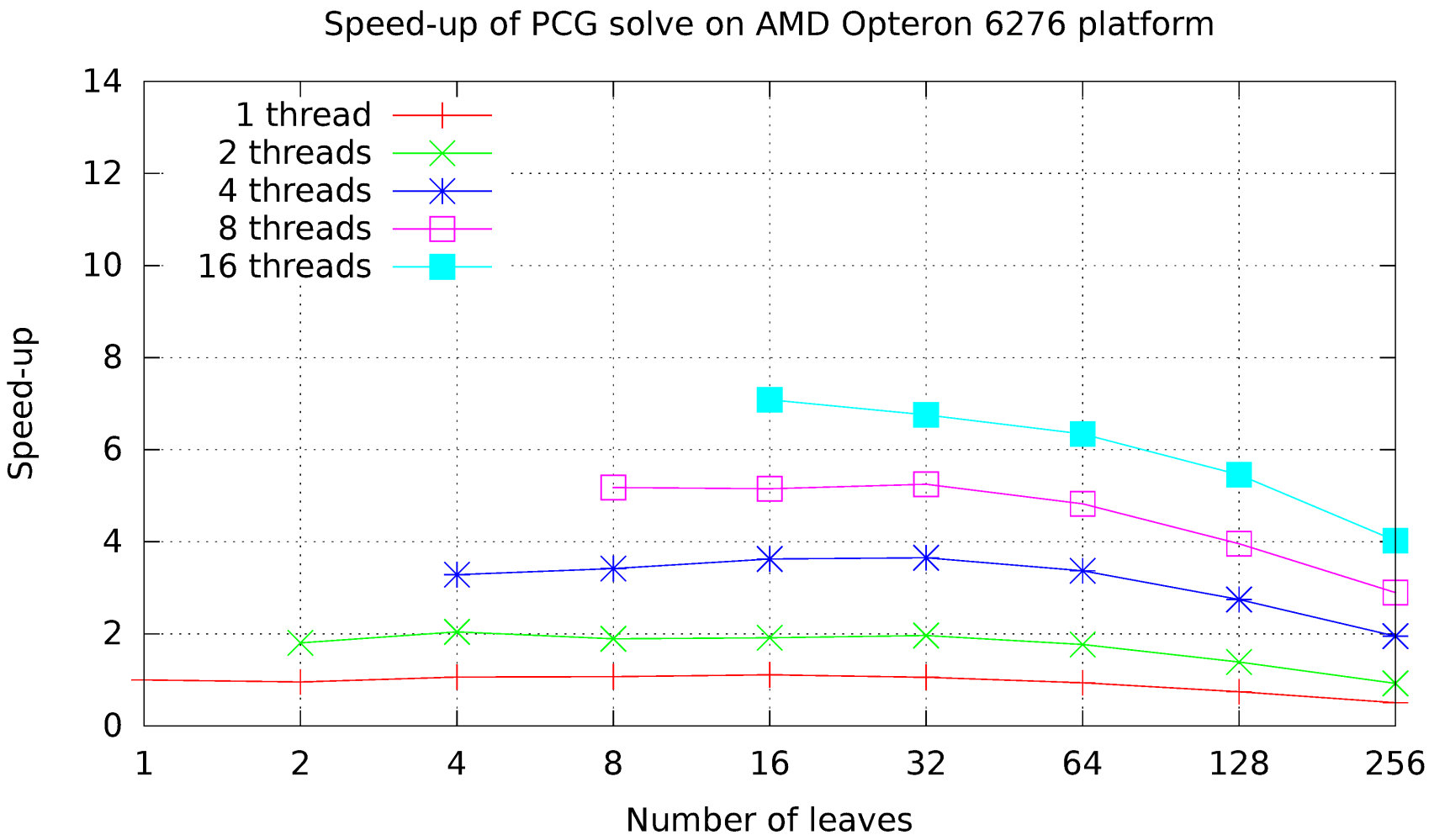


Merging small tasks:



OMPSS solver speed-up in AMD



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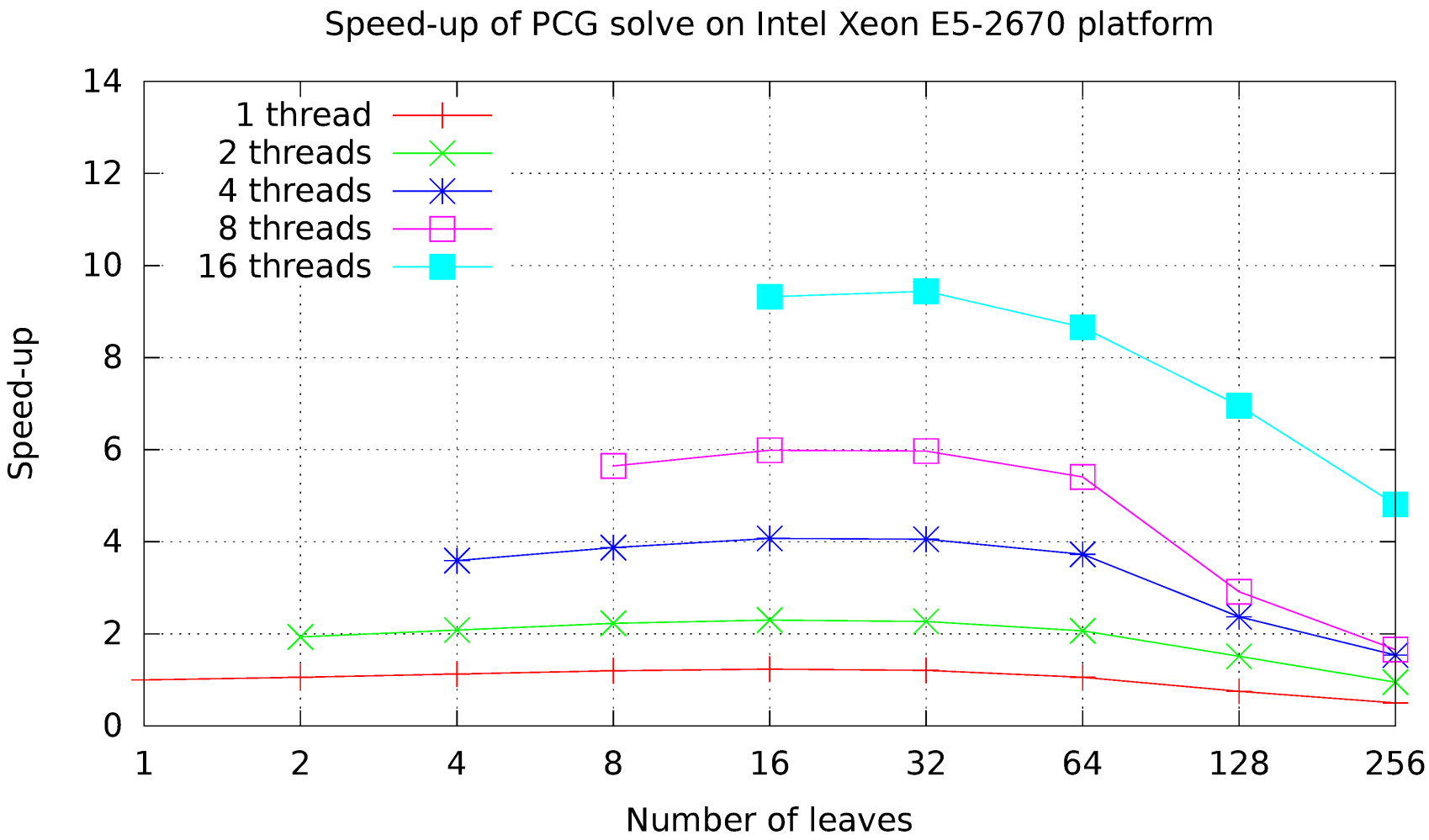
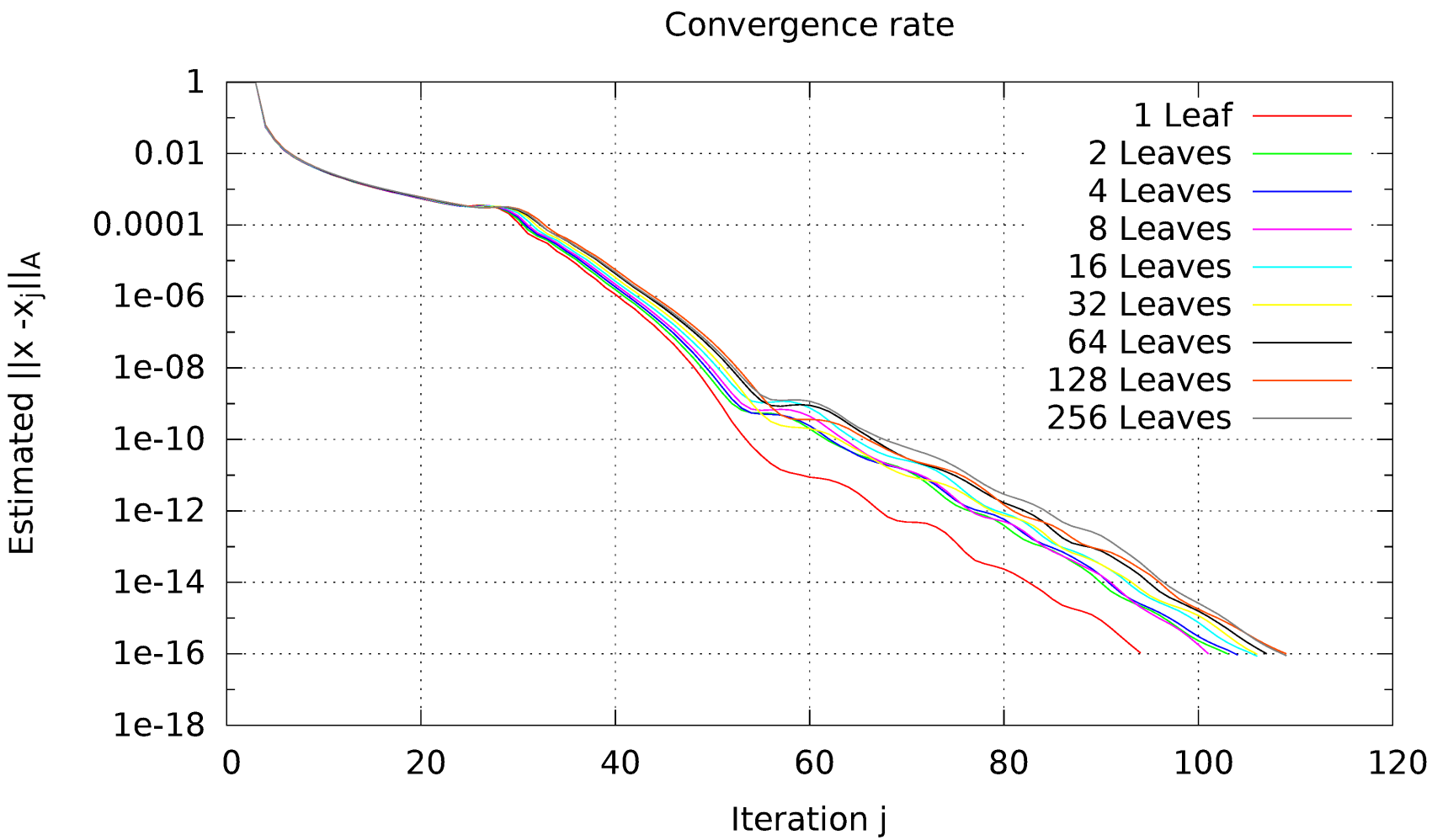


Parallel vs Sequential Solver

The conjugate gradient method minimizes at the jth step the energy norm of the error on the given j-dimensional Krylov subspace (Hestenes, 1952).

- We can estimate its value to analyze the behavior of each preconditioner.

Parallel vs Sequential Solver



Conclusions and Future work

The ILUPACK parallelization with OmpSs . . .

- Requires minor changes in the original code of ILUPACK
- Preserves the semantics of the method in ILUPACK
- Reduces the computation time by using the dynamic scheduling
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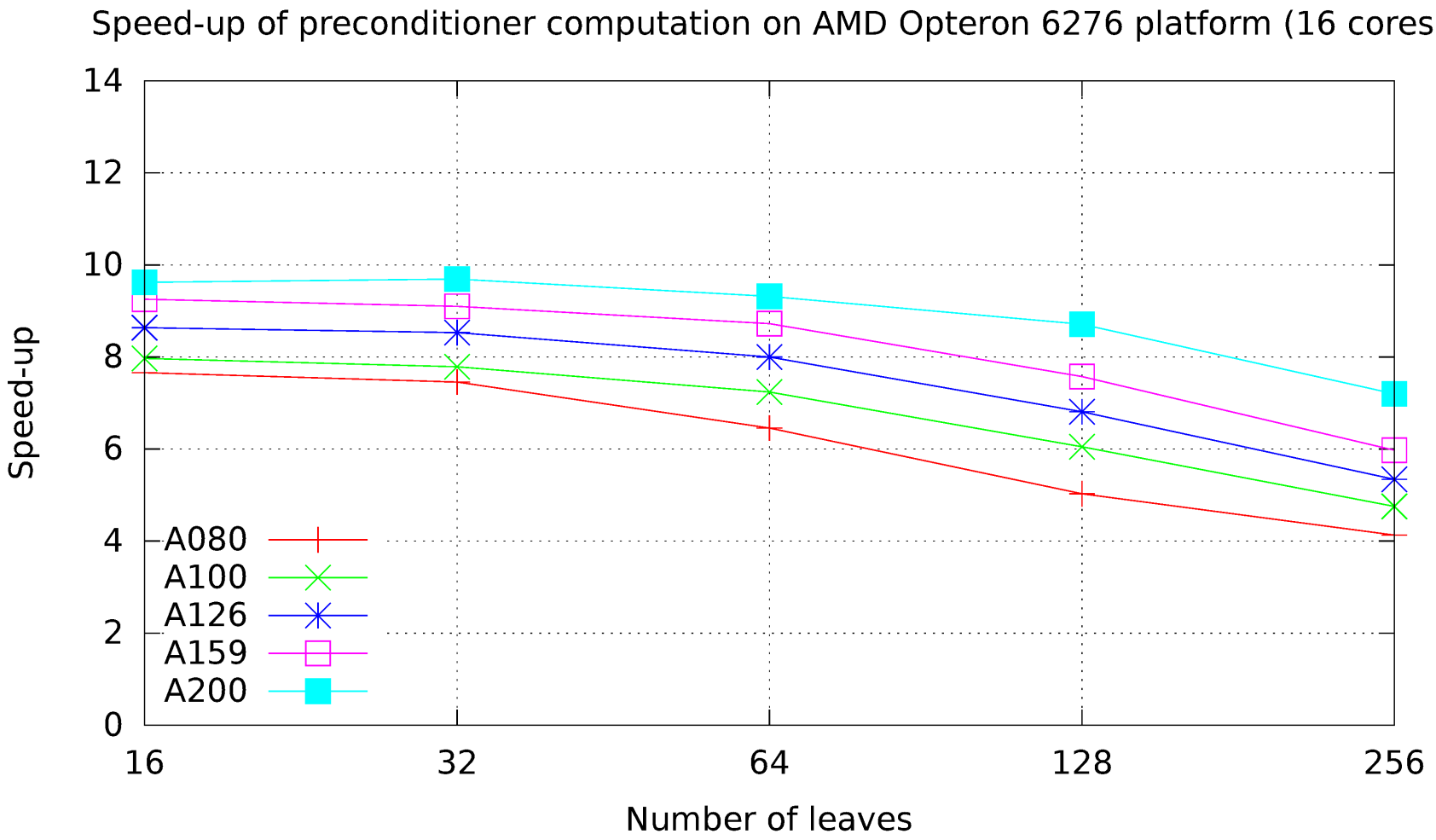
Future work . . .

- Investigate the effect of different options of the OmpSs runtime
  $\rightarrow$ Schedules that may favor data locality
  $\rightarrow$ Study how to improve the affinity in NUMA architectures
- Analyze the effect of power saving techniques on the performance
- Integrate energy saving mechanism into the OmpsS runtime scheduler
- Verify the distributed-memory and heterogeneous runtimes of OmpSs

Thanks for your attention !

Questions ?

OMPSS preconditioner computation speed-up in AMD



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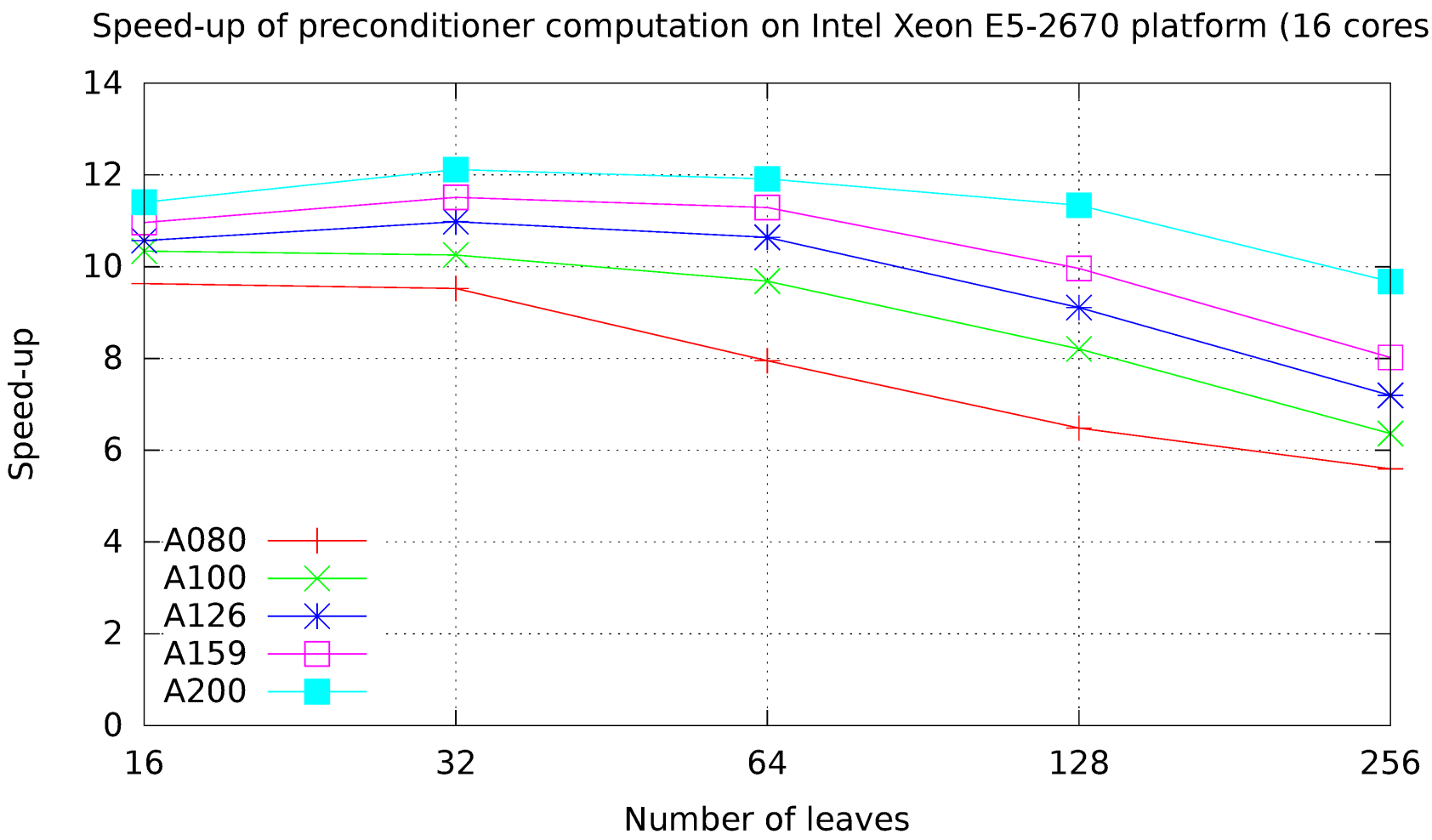

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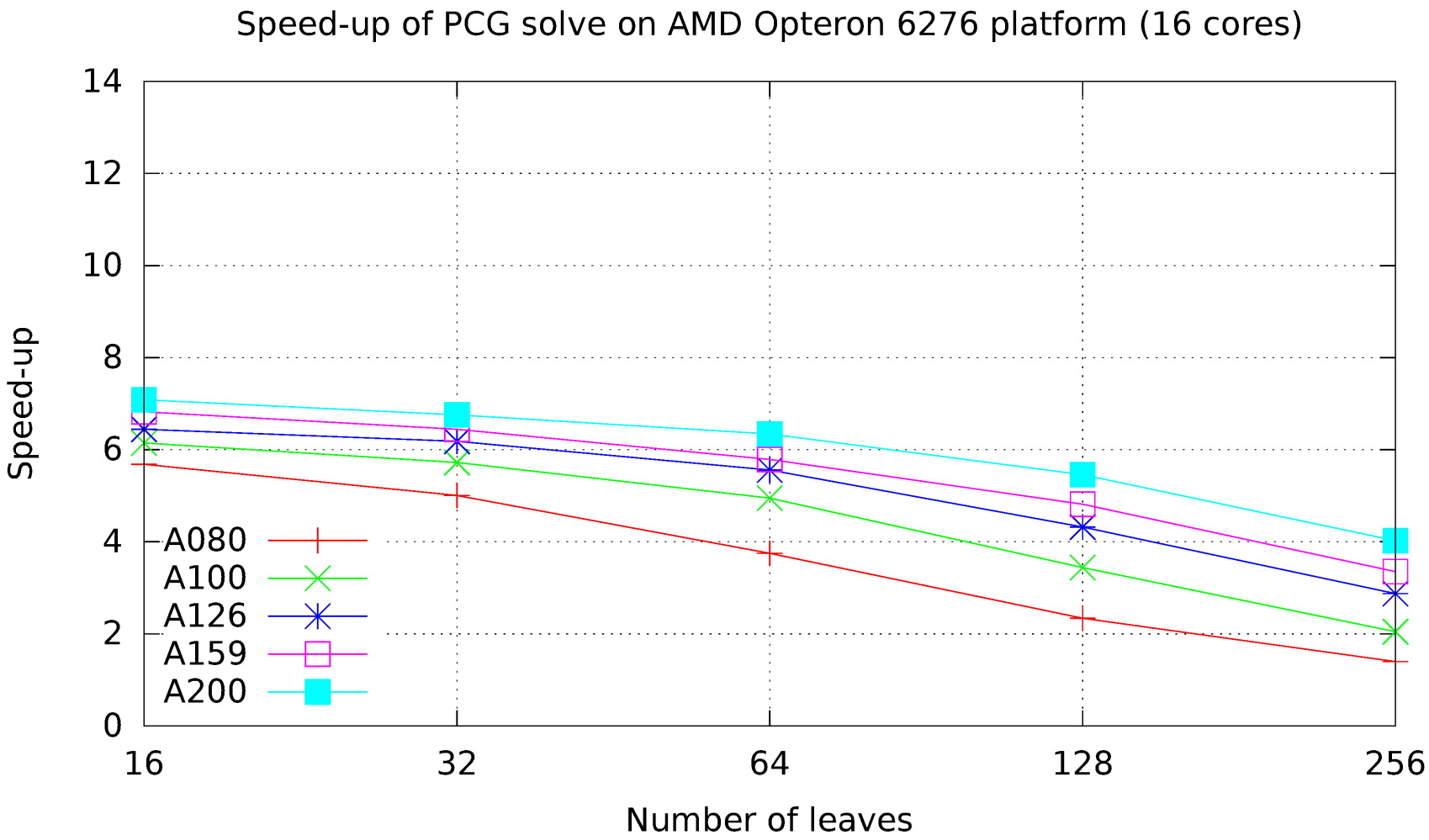

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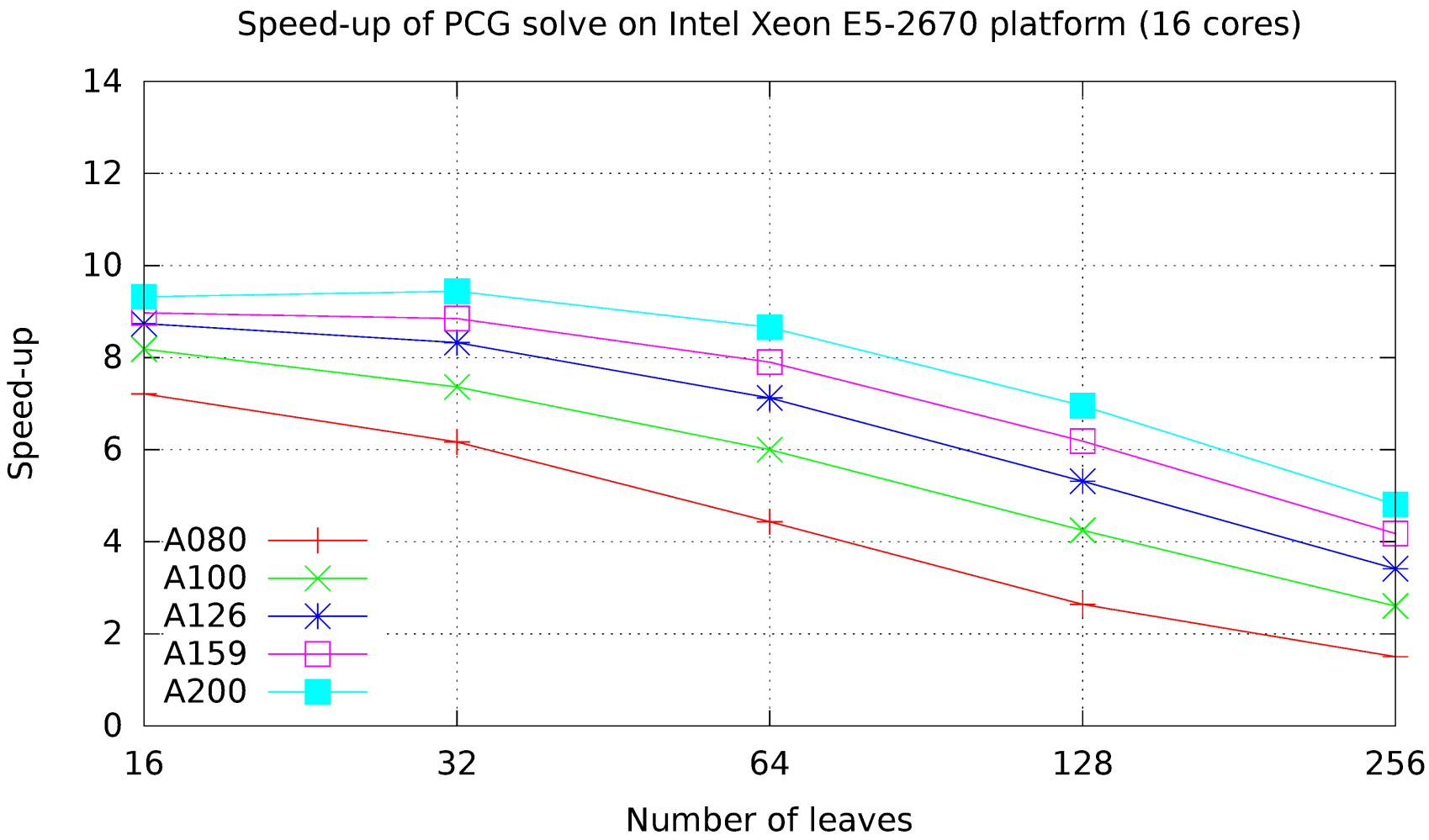

Parallel Preconditioned Conjugate Gradient

- The data is distributed on the leaves of the elimination tree:
  - Consistently: the information is replicated:

$$x_c = \begin{pmatrix} x_1 \\ x_2 \\ \hline x_3 \end{pmatrix} \rightarrow \left(\frac{x_1}{x_3} \right), \left(\frac{x_2}{x_3} \right).$$

  - Unconsistently: some information isn't replicated:

$$b_u = \begin{pmatrix} b_1 \\ b_2 \\ \hline b_3 \end{pmatrix} \rightarrow \left(\frac{b_1}{b_3^1} \right), \left(\frac{b_2}{b_3^2} \right), b_3^1 + b_3^2 = b_3.$$

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$$A_u = \left(\begin{array}{cc|c} A_{11} & 0 & A_{31}^T \\ 0 & A_{22} & A_{32}^T \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right) \rightarrow \left(\begin{array}{c|c} A_{11} & A_{31}^T \\ \hline A_{31} & A_{33}^1 \end{array}\right), \left(\begin{array}{c|c} A_{22} & A_{32}^T \\ \hline A_{32} & A_{33}^2 \end{array}\right), \quad A_{33}^1 + A_{33}^2 = A_{33}.$$

Parallel Preconditioned Conjugate Gradient

- The distribution defines how the computations are made:
  - The AXPY-like operations are made on similar operators.
  - The DOT-like operations are made on different operators:
  - The SPMV operations works as DOT-like operations:

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$$b_u^T * x_c = \left(\begin{array}{c} b_1 \\ \hline b_2 \\ \hline b_3 \end{array}\right)^T * \left(\begin{array}{c} x_1 \\ \hline x_2 \\ \hline x_3 \end{array}\right)$$

$$b_u^T * x_c = \left(\frac{b_1}{b_3^1}\right)^T * \left(\frac{x_1}{x_3}\right) + \left(\frac{b_2}{b_3^2}\right)^T * \left(\frac{x_2}{x_3}\right)$$

$$b_u^T * x_c = val_1 + val_2$$

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  - The SPMV operations works as DOT-like operations:

$$A_u * x_c = \left(\begin{array}{cc|c} A_{11} & 0 & A_{31}^T \\ 0 & A_{22} & A_{32}^T \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right) * \left(\begin{array}{c} x_1 \\ \hline x_2 \\ \hline x_3 \end{array}\right)$$

$$A_u * x_c = \left(\begin{array}{c|c} A_{11} & A_{31}^T \\ \hline A_{31} & A_{33}^1 \end{array}\right) * \left(\begin{array}{c} x_1 \\ \hline x_3 \end{array}\right) \oplus \left(\begin{array}{c|c} A_{22} & A_{32}^T \\ \hline A_{32} & A_{33}^2 \end{array}\right) * \left(\begin{array}{c} x_2 \\ \hline x_3 \end{array}\right)$$

$$A_u * x_c = \left(\begin{array}{c} v_1 \\ \hline v_3^1 \end{array}\right) \oplus \left(\begin{array}{c} v_2 \\ \hline v_3^2 \end{array}\right) = v_u$$

Parallel Preconditioned Conjugate Gradient

- The distribution defines how the computations are made:
  - The AXPY-like operations are made on similar operators.
  - The DOT-like operations are made on different operators:

$$b_u^T * x_c = \left(\begin{array}{c} b_1 \\ b_2 \\ \hline b_3 \end{array}\right)^T * \left(\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \end{array}\right) = val_1 + val_2 = val_u$$

  - The SPMV operations works as DOT-like operations:

$$A_u * x_c = \left(\begin{array}{cc|c} A_{11} & 0 & A_{31}^T \\ 0 & A_{22} & A_{32}^T \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right) * \left(\begin{array}{c} x_1 \\ x_2 \\ \hline x_3 \end{array}\right)$$

$$A_u * x_c = \left(\begin{array}{c} v_1 \\ 0 \\ \hline v_3^1 \end{array}\right) + \left(\begin{array}{c} 0 \\ v_2 \\ \hline v_3^2 \end{array}\right) = v_u$$

Parallel Preconditioned Conjugate Gradient

Initialize $k, x_0, r_0, z_0, d_0, \beta_0, \tau_0$
while $((k < k_{max})\&\&(\tau_k > \tau_{min}))$
$w_k := Ad_k$
$\gamma_k := d_k^T w_k$
$\rho_k := \beta_k / \gamma_k$
$x_{k+1} := x_k + \rho_k d_k$
$r_{k+1} := r_k - \rho_k w_k$
$z_{k+1} := M^{-1} r_{k+1}$
$\alpha_k := \beta_k$
$\beta_{k+1} := r_{k+1}^T z_{k+1}$
$\alpha_k := \beta_{k+1} / \alpha_k$
$d_{k+1} := z_{k+1} + \alpha_k d_k$
$\tau_{k+1} := ||r_{k+1}||_2$
$k := k + 1$
end while

At the beginning: A_u, x_c, b_u
Loop:
d_c, w_u
$\gamma_u \xrightarrow{reduction} \gamma_c$
ρ_c
x_c
r_u
$\xrightarrow{crossing} z_c$
α_c
$\beta_u \xrightarrow{reduction} \beta_c$
α_c

$\xrightarrow{crossing} r_c$, $\tau_u \xrightarrow{reduction} \tau_c$

Parallel Preconditioned Conjugate Gradient

Initialize $k, x, r, z, d, \beta, \tau$
while $((k < k_{max})\&\&(\tau > \tau_{min}))$
$w_u := A_u d_c$
$\gamma_c := d_c^T w_u$
$\rho_c := \beta_c / \gamma_c$
$x_c := x_c + \rho_c d_c$
$r_u := r_u - \rho_c w_u$
$\alpha_c := \beta_c$
$z_c := M^{-1} r_u$, $r_c \leftarrow r_u$
$\tau_c := sqrt(r_u^T r_c)$
$\beta_c := r_u^T z_c$
$\alpha_c := \beta_c / \alpha_c$
$d_c := z_c + \alpha_c d_c$
$k := k + 1$
end while

At the beginning: A_u, x_c, b_u
Loop:

$\gamma_u \xrightarrow{reduction} \gamma_c$

$\xrightarrow{crossing} z_c$, $\xrightarrow{crossing} r_c$
$\tau_u \xrightarrow{reduction} \tau_c$
$\beta_u \xrightarrow{reduction} \beta_c$