Parallel Model Reduction of Large Linear Descriptor Systems via Balanced Truncation

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Linear Systems

Linear time-invariant descriptor systems:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \quad t > 0, \quad x(0) = x^0, \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0, \end{aligned}$$

- n state-space variables, i.e., n is the order of the system,
- m inputs,
- p outputs,
- $A - \lambda E$ is stable and regular.

Corresponding TFM:

$$G(s) = C(sE - A)^{-1}B + D.$$

Model Reduction: Purpose

Given

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t), \quad t > 0, \quad x(0) = x^0, \\
y(t) &= Cx(t) + Du(t), \quad t \geq 0,
\end{aligned}
$$

find a reduced model

$$
\begin{aligned}
\hat{E}\dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}u(t), \quad t > 0, \quad \hat{x}(0) = \hat{x}^0, \\
\hat{y}(t) &= \hat{C}\hat{x}(t) + \hat{D}u(t), \quad t \geq 0,
\end{aligned}
$$

of order $r \ll n$ and output error

$$
y - \hat{y} = Gu - \hat{G}u = (G - \hat{G})u
$$

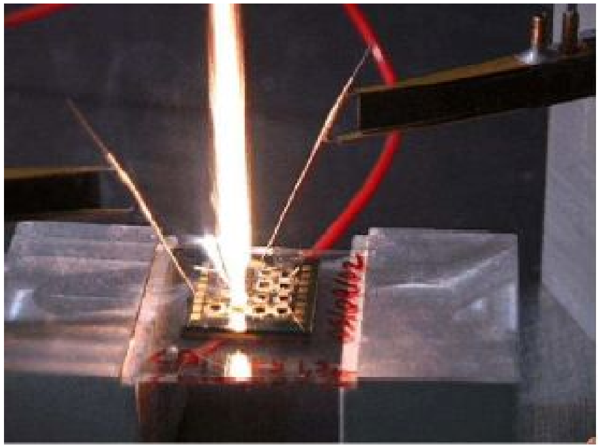
such that

$\|y - \hat{y}\|$ and $\|G - \hat{G}\|$ is "small"!

Model Reduction: MEMS Example

μ-thruster array [IMTEK (Univ. Freiburg)/EU project μ-pyros]

- Co-integration of solid fuel with silicon μ-machined system.
- Used for "nano-satellites" and gas generation.
- Design problem: reach the ignition temperature within the fuel without reaching the critical temperature at the neighbour μ-thrusters (boundary).



n from $4{,}257 - 79{,}171$ states, $p = 7$ outputs.

Model Reduction of Descriptor Systems

1) Decouple the system:

$$\begin{aligned} G(s) &= G_0(s) + G_\infty(s) \\ &= [C_0\, C_\infty] \left(s \begin{bmatrix} E_0 & 0 \\ 0 & E_\infty \end{bmatrix} - \begin{bmatrix} A_0 & 0 \\ 0 & A_\infty \end{bmatrix} \right)^{-1} \begin{bmatrix} B_0 \\ B_\infty \end{bmatrix} + D, \end{aligned}$$

where $\Lambda(A_\infty - \lambda E_\infty)$ contains the infinite poles of the system.

In model reduction, the order of $(A_\infty - \lambda E_\infty, B_\infty, C_\infty, 0)$ is usually small.

2) Approximate $G_0(s)$ by $\hat{G}_0(s)$ so that

$$\hat{G}(s) = \hat{G}_0(s) + G_\infty(s).$$

Decoupling TFMs of Descriptor Systems

1. Compute orthogonal $Q, Z \in \mathbb{R}^{n \times n}$ s.t.

$$Q^T(A - \lambda E)Z = \begin{bmatrix} A_0 & W_A \\ 0 & A_\infty \end{bmatrix} - \lambda \begin{bmatrix} E_0 & W_E \\ 0 & E_\infty \end{bmatrix}.$$

   1.1. Use spectral projectors such as the matrix sign/disk functions.

2. Solve the generalized Sylvester equation

$$A_0 Y + X A_\infty + W_A = 0, \quad E_0 Y + X E_\infty + W_E = 0.$$

3. Thus,

$$\hat{A} - \lambda \hat{E} := \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} Q^T(A - \lambda E)Z \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ 0 & A_\infty \end{bmatrix} - \lambda \begin{bmatrix} E_0 & 0 \\ 0 & E_\infty \end{bmatrix}.$$

BT Model Reduction of Descriptor Systems

1. Solve the "coupled" generalized Lyapunov matrix equations

$$\begin{aligned} A_0 W_c E_0^T + E_0 W_c A_0^T + B_0 B_0^T &= 0, \\ A_0^T \hat{W}_o E_0 + E_0^T \hat{W}_o A_0 + C_0^T C_0 &= 0, \quad W_o = E_0^T \hat{W}_o E_0, \end{aligned}$$

   for S, R such that $W_c = S^T S$, $W_o = R^T R$.

2. Compute

$$S R^T = U \Sigma V^T = [\, U_1 \; U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}, \quad \Sigma_1 \in \mathbb{R}^{r \times r}.$$

3. In the square-root (SR) method:

$$(\hat{A}_0 - \lambda \hat{E}_0, \hat{B}_0, \hat{C}_0, \hat{D}_0) = (L(A_0 - \lambda E_0)T, L B_0, C T_0, D),$$

   with

$$L = \Sigma_1^{-1/2} V_1^T R {E_0}^{-1} \quad \text{and} \quad T = S^T U_1 \Sigma_1^{-1/2}.$$

Model Reduction of Descriptor Systems

Given a large system $(A - \lambda E, B, C, D)$ with $m, p \ll n \ldots$

How do we solve the previous numerical problems?

1. Decouple the TFM.
2. Solve the coupled generalized Lyapunov equations.
3. Compute SVD of matrix product and apply the SRBT formulae.

1. Decouple the TFM

Malyshev's iteration provides the appropriate spectral projectors:

$$A_0 \leftarrow A, \;\; B_0 \leftarrow B, \;\; R_0 \leftarrow 0,$$

repeat

Compute the QR factorization

$$\begin{bmatrix} A_j \\ -B_j \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} R_{j+1} \\ 0 \end{bmatrix}$$

$$A_{j+1} \leftarrow U_{12}^T A_j,$$
$$B_{j+1} \leftarrow U_{22}^T B_j,$$

- Composed of usual matrix operations from LA: QR fact., matrix product.
- Further improved by Bai/Demmel/Gu and Sun/Quintana.

2. Solution of generalized Lyapunov equations

- Traditional methods reduce $A - \lambda E$ to generalized (real) Schur form and solve the resulting "triangular" equation (Bartels, Stewart, 72; Hammarling, 82):
  - Produce dense Cholesky factors of order n.
  - Requires a parallel version of the QZ algorithm.
- Sign function methods compute an spectral projector (Roberts, 71):
  - More reliable if S and R are numerically singular.
  - Reduced form is better conditioned.
  - Also more efficient as, usually, $\mathrm{rank}\,(S)$, $\mathrm{rank}\,(R) \ll n \ldots$
  - Highly parallel!

2. Solution of generalized Lyapunov equations (Cont.)

We employ a variant of the Newton iteration for the sign function:

$$A_0 \leftarrow A, \quad R_0 \leftarrow C$$

repeat

$$A_{j+1} \leftarrow \frac{1}{\sqrt{2}}\left(A_j + E A_j^{-1} E\right)$$

Compute the RRQR decomposition

$$\frac{1}{\sqrt{2}} \begin{bmatrix} R_j \\ R_j A_j^{-1} E \end{bmatrix} = Q_r \begin{bmatrix} R_r \\ 0 \end{bmatrix} \Pi_r$$

$$R_{j+1} \leftarrow (R_r \Pi_r)^T$$

- Composed of usual matrix operations from LA, including QRP Fact.
- On convergence, provides a full-rank factor $\hat{R}$ with $l \ll n$ columns!

3. SVD and application of SRBT formulae

Replace the Cholesky factors by their low-rank approximations in

$$SR^T \approx \hat{S}^T \hat{R} = U \Sigma V^T.$$

Implementation:

- Accuracy can be enhanced by computing the SVD withouth computing explicitly the product but it is difficult.

4. Application of SRBT formulae

Let

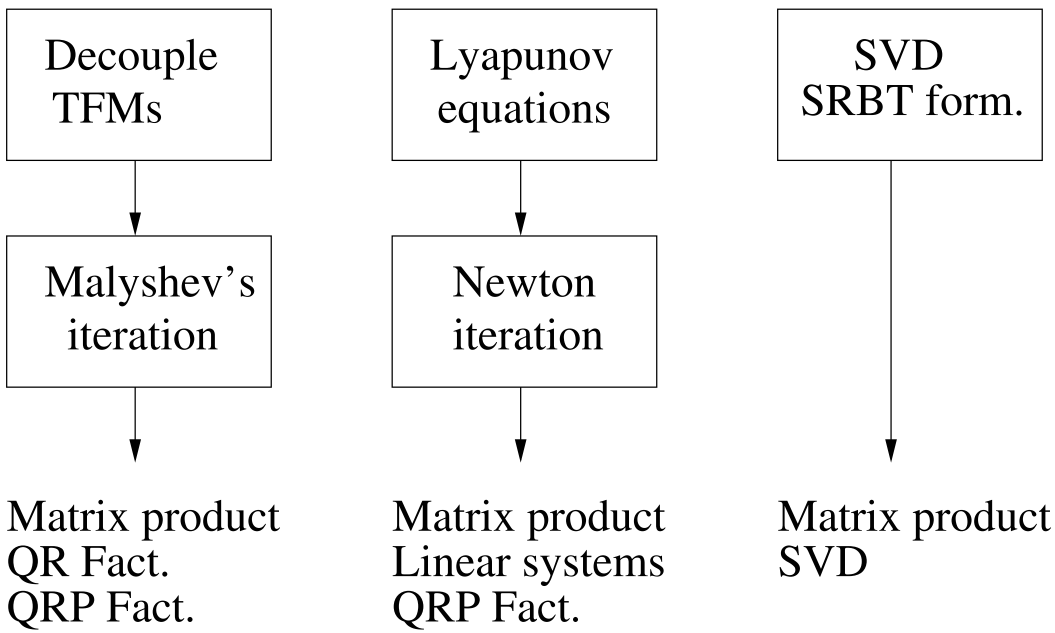
$$L = \Sigma_1^{-1/2} V_1^T \hat{R}^T, \quad T = \hat{S} U_1 \Sigma_1^{-1/2},$$

then

$$\begin{aligned} \hat{E} &= LET, & \hat{A} &= LAT, \\ \hat{B} &= LB, & \hat{C} &= CT \end{aligned}$$

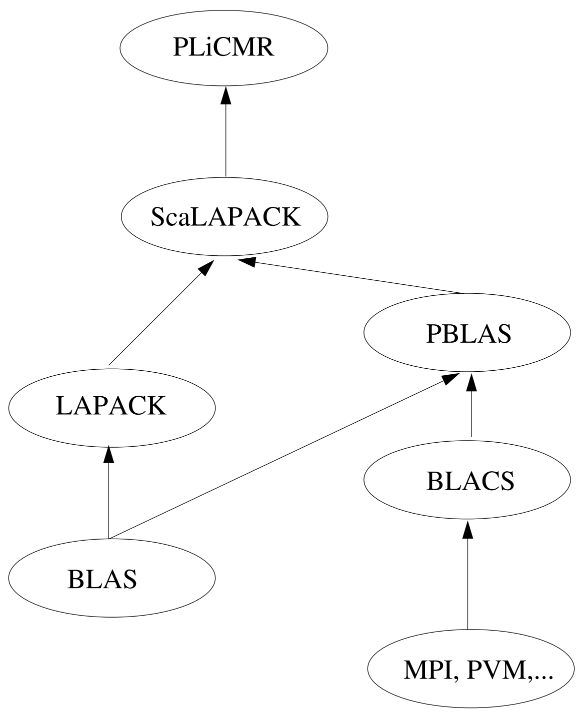
Parallelization

Variety of LA operations:



Parallelization (Cont.)

Use dense parallel LA libraries:



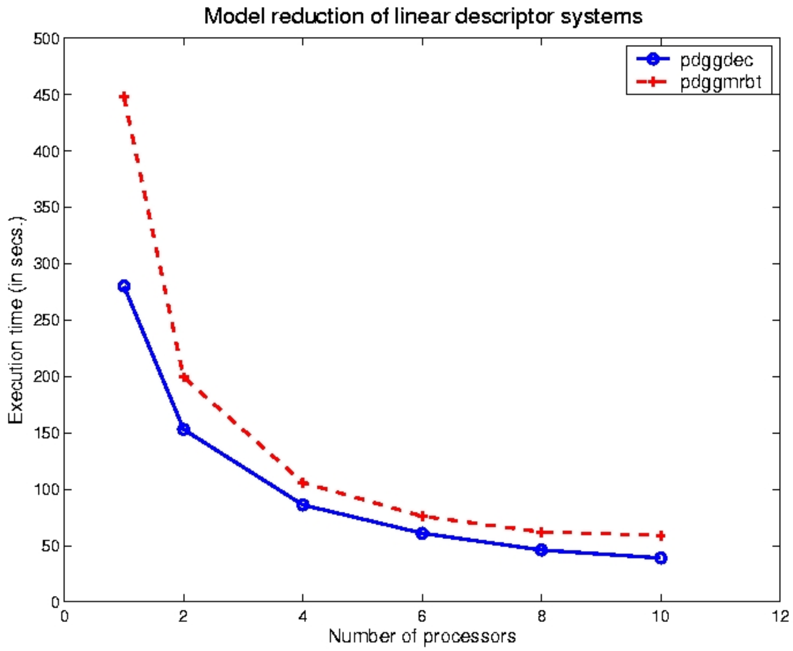
Experimental Results

Computing facility:

- 32 nodes $\times$ 2 Intel Pentium Xeon@2.4GHz, 1GB RAM.
- Myrinet interconnection switches, 2Gbps peak bandwidth.
- IEEE double precision arithmetic.

Experimental Results (Cont. I)

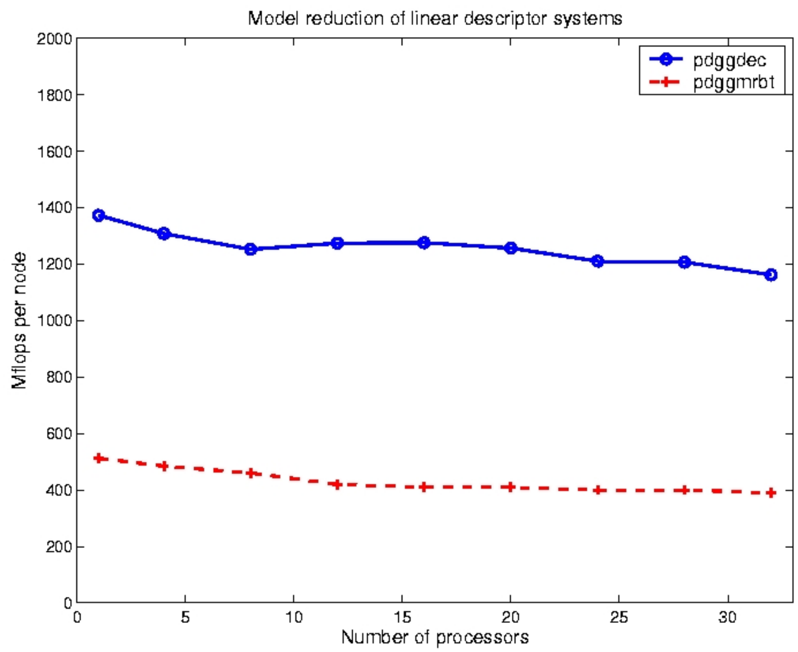
Efficiency ($n = 1800$, $m = p = 180$):



- Speed-up of 3.25/4.25 (!) obtained for `pdggdec`/`pdggmrbt` on 4 nodes.
- Using $>$8–10 nodes is useless for such a "small" problem.

Experimental Results (Cont. II)

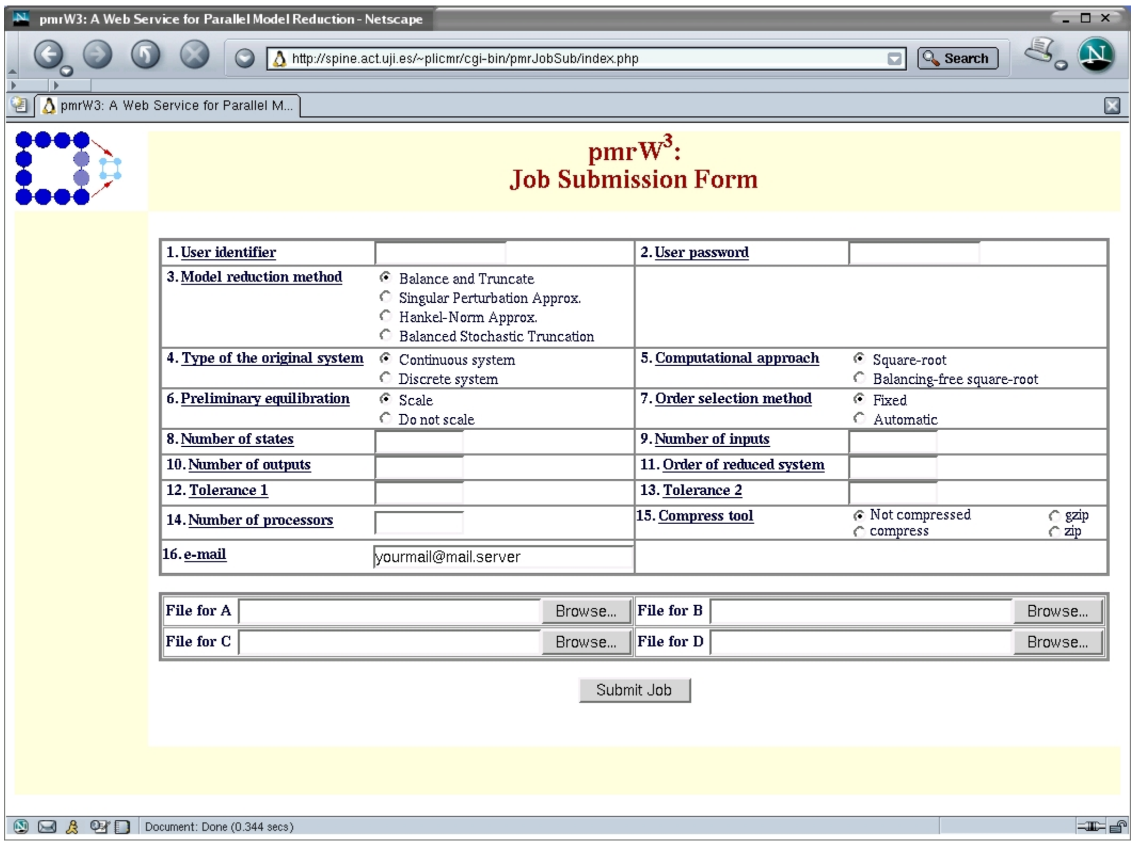
Scalability ($n/\sqrt{n_p} = 1800$, $m/\sqrt{n_p} = p/\sqrt{n_p} = 180$):



- Excellent scalability, at least as far as 32 nodes.
- Slower performance of `pdggmrbt` is due to intensive use of QRP Fact.

Friendly Access

Visit `http://spine.act.uji.es/ plicmr/`



Concluding Remarks

- Existing serial libraries are not powerful enough:

  SLICOT $\rightarrow \mathcal{O}(10^3)$.

- Parallel SRBT algorithms in PLiCMR (PSLICOT) allow reduction of descriptor systems with $\mathcal{O}(10^4)$ states.

- Efficacy depends on parallelism of underlying parallel libraries and, in general, is good.

- Please, contact us if you have any large stable systems to reduce

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